Opti 501 — 1st Midterm Solutions (10/1/2020) — Time: 75 minutes

Problem 1) $\boldsymbol{A}(\boldsymbol{r},t)\times\boldsymbol{B}(\boldsymbol{r},t)=(\boldsymbol{A}_0\times\boldsymbol{B}_0)\exp\{\mathrm{i}[(\boldsymbol{k}_a+\boldsymbol{k}_b)\cdot\boldsymbol{r}-(\omega_a+\omega_b)t]\}$

$$\rightarrow\quad \boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B})=\mathrm{i}(\boldsymbol{k}_a+\boldsymbol{k}_b)\cdot(\boldsymbol{A}_0\times\boldsymbol{B}_0)\exp\{\mathrm{i}[(\boldsymbol{k}_a+\boldsymbol{k}_b)\cdot\boldsymbol{r}-(\omega_a+\omega_b)t]\}.$$

$$\begin{aligned}\boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})&=\boldsymbol{A}_0\exp[\mathrm{i}(\boldsymbol{k}_a\cdot\boldsymbol{r}-\omega_a t)]\cdot\{\mathrm{i}\boldsymbol{k}_b\times\boldsymbol{B}_0\exp[\mathrm{i}(\boldsymbol{k}_b\cdot\boldsymbol{r}-\omega_b t)]\}\\&=\mathrm{i}\boldsymbol{A}_0\cdot(\boldsymbol{k}_b\times\boldsymbol{B}_0)\exp\{\mathrm{i}[(\boldsymbol{k}_a+\boldsymbol{k}_b)\cdot\boldsymbol{r}-(\omega_a+\omega_b)t]\}\\&=\mathrm{i}\boldsymbol{k}_b\cdot(\boldsymbol{B}_0\times\boldsymbol{A}_0)\exp\{\mathrm{i}[(\boldsymbol{k}_a+\boldsymbol{k}_b)\cdot\boldsymbol{r}-(\omega_a+\omega_b)t]\}.\end{aligned}$$

$$\begin{aligned}\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})&=\boldsymbol{B}_0\exp[\mathrm{i}(\boldsymbol{k}_b\cdot\boldsymbol{r}-\omega_b t)]\cdot\{\mathrm{i}\boldsymbol{k}_a\times\boldsymbol{A}_0\exp[\mathrm{i}(\boldsymbol{k}_a\cdot\boldsymbol{r}-\omega_a t)]\}\\&=\mathrm{i}\boldsymbol{B}_0\cdot(\boldsymbol{k}_a\times\boldsymbol{A}_0)\exp\{\mathrm{i}[(\boldsymbol{k}_a+\boldsymbol{k}_b)\cdot\boldsymbol{r}-(\omega_a+\omega_b)t]\}\\&=\mathrm{i}\boldsymbol{k}_a\cdot(\boldsymbol{A}_0\times\boldsymbol{B}_0)\exp\{\mathrm{i}[(\boldsymbol{k}_a+\boldsymbol{k}_b)\cdot\boldsymbol{r}-(\omega_a+\omega_b)t]\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})-\boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})&=[\mathrm{i}\boldsymbol{k}_a\cdot(\boldsymbol{A}_0\times\boldsymbol{B}_0)-\mathrm{i}\boldsymbol{k}_b\cdot(\boldsymbol{B}_0\times\boldsymbol{A}_0)]\exp\{\mathrm{i}[(\boldsymbol{k}_a+\boldsymbol{k}_b)\cdot\boldsymbol{r}-(\omega_a+\omega_b)t]\}\\&=\mathrm{i}(\boldsymbol{k}_a+\boldsymbol{k}_b)\cdot(\boldsymbol{A}_0\times\boldsymbol{B}_0)\exp\{\mathrm{i}[(\boldsymbol{k}_a+\boldsymbol{k}_b)\cdot\boldsymbol{r}-(\omega_a+\omega_b)t]\}\\&=\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}).\end{aligned}$$

Problem 2) a) Differentiation with respect to t can move in and out of the divergence operator:

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{bound}}^{(\text{e})}+\partial\rho_{\text{bound}}^{(\text{e})}/\partial t=\boldsymbol{\nabla}\cdot(\partial\boldsymbol{P}/\partial t)+\partial(-\boldsymbol{\nabla}\cdot\boldsymbol{P})/\partial t=\boldsymbol{\nabla}\cdot(\partial\boldsymbol{P}/\partial t)-\boldsymbol{\nabla}\cdot(\partial\boldsymbol{P}/\partial t)=0.$$

b) Considering that the divergence of the curl of any vector field is zero, we will have

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{bound}}^{(\text{e})}+\partial\rho_{\text{bound}}^{(\text{e})}/\partial t=\boldsymbol{\nabla}\cdot(\mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M})+0=\mu_0^{-1}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{M})=0.$$

Problem 3) a) $\boldsymbol{E}(\boldsymbol{r},t)=E_0\hat{\boldsymbol{z}}\cos(k_x x)\cos(k_y y-\omega t)\quad\rightarrow\quad\boldsymbol{\nabla}\cdot\boldsymbol{E}=\partial E_z/\partial z=0.$

b) On the left- and right-hand sides of the cavity (i.e., at $x=\pm L_x/2$), the tangential component E_z of the E-field must vanish. Therefore,

$$\cos(\pm k_x L_x/2)=0\quad\rightarrow\quad k_x L_x/2=(\underbrace{n}_{\text{arbitrary integer }(0,1,2,\cdots)}+½)\pi\quad\rightarrow\quad L_x=(2n+1)\pi/k_x.$$

c) At the upper and lower surfaces of the cavity, the perpendicular E-field is E_z. Consequently,

$$\sigma_s(x,y,\pm½L_z,t)=\mp\varepsilon_0E_z(x,y,\pm½L_z,t)=\mp\varepsilon_0E_0\cos(k_x x)\cos(k_y y-\omega t).$$

d) $$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{E}&=(\partial E_z/\partial y)\hat{\boldsymbol{x}}-(\partial E_z/\partial x)\hat{\boldsymbol{y}}\\&=-E_0k_y\cos(k_x x)\sin(k_y y-\omega t)\,\hat{\boldsymbol{x}}+E_0k_x\sin(k_x x)\cos(k_y y-\omega t)\,\hat{\boldsymbol{y}}=-\partial\boldsymbol{B}/\partial t.\end{aligned}$$

Integrating the above expression of $\partial\boldsymbol{B}/\partial t$ with respect to t, and ignoring the constant of integration, we find

$$\boldsymbol{B}(\boldsymbol{r},t)=(E_0k_y/\omega)\cos(k_x x)\cos(k_y y-\omega t)\,\hat{\boldsymbol{x}}+(E_0k_x/\omega)\sin(k_x x)\sin(k_y y-\omega t)\,\hat{\boldsymbol{y}}.$$

e) $$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{B}&=\partial B_x/\partial x+\partial B_y/\partial y\\&=-(E_0k_xk_y/\omega)\sin(k_x x)\cos(k_y y-\omega t)+(E_0k_xk_y/\omega)\sin(k_x x)\cos(k_y y-\omega t)=0.\end{aligned}$$

f) The x-component of the B-field is perpendicular to the cavity walls on the right- and left-hand-sides. For B_x found in part (d) to vanish at $x = \pm L_x/2$, we must have $\cos(\pm k_x L_x/2) = 0$. This, however, is the same condition as found in part (b). Consequently, $L_x = (2n+1)\pi/k_x$ satisfies both boundary conditions for $\boldsymbol{E}_\parallel$ and $\boldsymbol{B}_\perp$ on the right- and left-hand-side walls.

g) The surface current-density $\boldsymbol{J}_s$ at the top and bottom facets of the waveguide equals the tangential H-field at the corresponding surface, with the caveat that the direction of the current is perpendicular to that of the tangential H-field and satisfies the right-hand rule. We thus have

$$\boldsymbol{J}_s(x, y, \pm\tfrac{1}{2}L_z, t) = \pm H_y(x, y, \pm\tfrac{1}{2}L_z, t)\hat{\boldsymbol{x}} \mp H_x(x, y, \pm\tfrac{1}{2}L_z, t)\hat{\boldsymbol{y}}$$
$$= \pm(E_0 k_x/\mu_0\omega)\sin(k_x x)\sin(k_y y - \omega t)\,\hat{\boldsymbol{x}} \mp (E_0 k_y/\mu_0\omega)\cos(k_x x)\cos(k_y y - \omega t)\,\hat{\boldsymbol{y}}.$$

Digression: In addition, there exist surface currents on the left and right sidewalls, as follows:

$$\boldsymbol{J}_s(\pm L_x/2, y, z, t) = -(E_0 k_x/\mu_0\omega)\sin(k_x L_x/2)\sin(k_y y - \omega t)\,\hat{\boldsymbol{z}}.$$

This current connects those flowing in and out of the upper and lower walls at the four corners of the waveguide.

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h) $\boldsymbol{\nabla} \times \boldsymbol{H} = \cancelto{0}{\boldsymbol{J}_{\text{free}}} + \partial \boldsymbol{D}/\partial t \quad \rightarrow \quad \partial H_y/\partial x - \partial H_x/\partial y = \varepsilon_0\, \partial E_z/\partial t$

$$\rightarrow \quad (E_0 k_x^2/\mu_0\omega)\cos(k_x x)\sin(k_y y - \omega t) + (E_0 k_y^2/\mu_0\omega)\cos(k_x x)\sin(k_y y - \omega t)$$
$$= \varepsilon_0 E_0 \omega \cos(k_x x)\sin(k_y y - \omega t)$$
$$\rightarrow \quad (k_x^2 + k_y^2)/(\mu_0\omega) = \varepsilon_0\omega \quad \rightarrow \quad k_x^2 + k_y^2 = (\omega/c)^2.$$

i) $\boldsymbol{\nabla} \cdot \boldsymbol{J}_s + \partial\sigma_s/\partial t = 0 \quad \rightarrow \quad (\partial J_{sx}/\partial x) + (\partial J_{sy}/\partial y) + (\partial\sigma_s/\partial t) = 0$

$$\rightarrow \quad \pm(E_0 k_x^2/\mu_0\omega)\cos(k_x x)\sin(k_y y - \omega t) \pm (E_0 k_y^2/\mu_0\omega)\cos(k_x x)\sin(k_y y - \omega t)$$
$$\mp\varepsilon_0\omega E_0 \cos(k_x x)\sin(k_y y - \omega t) = 0$$
$$\rightarrow \quad \pm\left[(k_x^2 + k_y^2)/(\mu_0\omega) - \varepsilon_0\omega\right] = 0 \ \rightarrow\ \cancel{\varepsilon_0\omega} - \cancel{\varepsilon_0\omega} = 0 \ \rightarrow\ \text{continuity equation is satisfied.}$$

j) $\boldsymbol{S}(\boldsymbol{r}, t) = \boldsymbol{E} \times \boldsymbol{H} = E_0 \cos(k_x x)\cos(k_y y - \omega t)\,\hat{\boldsymbol{z}}$

$$\times \left(\frac{E_0}{\mu_0\omega}\right)[k_y \cos(k_x x)\cos(k_y y - \omega t)\,\hat{\boldsymbol{x}} + k_x \sin(k_x x)\sin(k_y y - \omega t)\,\hat{\boldsymbol{y}}]$$
$$= \left(\frac{E_0^2}{\mu_0\omega}\right)\{k_y \cos^2(k_x x)\cos^2(k_y y - \omega t)\,\hat{\boldsymbol{y}} - \tfrac{1}{4}k_x \sin(2k_x x)\sin[2(k_y y - \omega t)]\,\hat{\boldsymbol{x}}\}.$$

In the wave-propagation direction (i.e., along the y-axis), the flow of energy is forward and proportional to k_y. Considering that $\cos^2(k_y y - \omega t) = \tfrac{1}{2} + \tfrac{1}{2}\cos[2(k_y y - \omega t)]$, the *time-averaged* rate of energy flow along $\hat{\boldsymbol{y}}$ is $(k_y E_0^2/2\mu_0\omega)\cos^2(k_x x)$. The remaining term, namely $(k_y E_0^2/2\mu_0\omega)\cos^2(k_x x)\cos[2(k_y y - \omega t)]$, indicates a forward/backward motion of energy along $\pm\hat{\boldsymbol{y}}$ (as a function of time), but no net flow in either direction. Similarly, the x-component of the Poynting vector, $S_x(\boldsymbol{r}, t) = -(k_x E_0^2/4\mu_0\omega)\sin(2k_x x)\sin[2(k_y y - \omega t)]$, indicates a sideways motion of the energy along the x-axis, although no net energy flows in either $+\hat{\boldsymbol{x}}$ or $-\hat{\boldsymbol{x}}$ direction.
