Opti 501 **1$^{\text{st}}$ Midterm Solutions** (10/3/2019) **Time: 75 minutes**

Problem 1) Invoking the product rule of differentiation, namely, $(fg)' = f'g + fg'$, we write

$$\boldsymbol{\nabla}\cdot(\psi\boldsymbol{A}) = \frac{\partial(r^2\psi A_r)}{r^2\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\psi A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\psi A_\varphi)}{\partial\varphi}$$

$$= \frac{1}{r^2}\left[\psi\frac{\partial(r^2 A_r)}{\partial r} + r^2 A_r\frac{\partial\psi}{\partial r}\right] + \frac{1}{r\sin\theta}\left[\psi\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \sin\theta\, A_\theta\frac{\partial\psi}{\partial\theta}\right] + \frac{1}{r\sin\theta}\left(\psi\frac{\partial A_\varphi}{\partial\varphi} + A_\varphi\frac{\partial\psi}{\partial\varphi}\right)$$

$$= \psi\left[\frac{\partial(r^2 A_r)}{r^2\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}\right] + \left(A_r\frac{\partial\psi}{\partial r} + A_\theta\frac{\partial\psi}{r\partial\theta} + A_\varphi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\right)$$

$$= \psi\boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\nabla}\psi.$$

Problem 2)

a) In the free-space region between the two mirrors, $\rho_{\text{free}} = 0$, $\boldsymbol{J}_{\text{free}} = 0$, $\boldsymbol{P} = 0$, and $\boldsymbol{M} = 0$. Maxwell's equations thus reduce to

(i) $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$; (ii) $\boldsymbol{\nabla}\times\boldsymbol{H} = \varepsilon_0\partial_t\boldsymbol{E}$; (iii) $\boldsymbol{\nabla}\times\boldsymbol{E} = -\mu_0\partial_t\boldsymbol{H}$; (iv) $\boldsymbol{\nabla}\cdot\boldsymbol{H} = 0$.

To confirm that the 1$^{\text{st}}$ equation is satisfied, we write

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \partial_x E_x + \cancel{\partial_y E_y} + \cancel{\partial_z E_z} = \partial_x E_x = E_0\partial_x[\sin(\omega z/c)\sin(\omega t)] = 0.$$

As for the second equation, we have

$$\boldsymbol{\nabla}\times\boldsymbol{H} = \cancel{(\partial_x H_y)}\hat{\boldsymbol{z}} - (\partial_z H_y)\hat{\boldsymbol{x}} = -(\partial_z H_y)\hat{\boldsymbol{x}} = -(E_0/Z_0)\partial_z[\cos(\omega z/c)\cos(\omega t)]\hat{\boldsymbol{x}}$$

$$= (E_0/Z_0)(\omega/c)\sin(\omega z/c)\cos(\omega t)\,\hat{\boldsymbol{x}} = \varepsilon_0 E_0\omega\sin(\omega z/c)\cos(\omega t)\,\hat{\boldsymbol{x}}.$$

$$\partial_t\boldsymbol{D} = \varepsilon_0\partial_t\boldsymbol{E} = \varepsilon_0\partial_t[E_0\sin(\omega z/c)\sin(\omega t)]\hat{\boldsymbol{x}} = \varepsilon_0 E_0\omega\sin(\omega z/c)\cos(\omega t)\,\hat{\boldsymbol{x}}.$$

Clearly, the 2$^{\text{nd}}$ equation is also satisfied. To verify the 3$^{\text{rd}}$ equation, we write

$$\boldsymbol{\nabla}\times\boldsymbol{E} = (\partial_z E_x)\hat{\boldsymbol{y}} - \cancel{(\partial_y E_x)}\hat{\boldsymbol{z}} = (\partial_z E_x)\hat{\boldsymbol{y}}$$

$$= E_0\partial_z[\sin(\omega z/c)\sin(\omega t)]\hat{\boldsymbol{y}} = E_0(\omega/c)\cos(\omega z/c)\sin(\omega t)\,\hat{\boldsymbol{y}}.$$

$$\partial_t\boldsymbol{B} = \mu_0\partial_t\boldsymbol{H} = \mu_0\partial_t[(E_0/Z_0)\cos(\omega z/c)\cos(\omega t)]\hat{\boldsymbol{y}}$$

$$= -\mu_0(E_0/Z_0)\omega\cos(\omega z/c)\sin(\omega t)\,\hat{\boldsymbol{y}} = -E_0(\omega/c)\cos(\omega z/c)\sin(\omega t)\,\hat{\boldsymbol{y}}.$$

The above equations confirm that $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B}$. Finally, the 4$^{\text{th}}$ equation is verified as follows:

$$\boldsymbol{\nabla}\cdot\boldsymbol{H} = \cancel{\partial_x H_x} + \partial_y H_y + \cancel{\partial_z H_z} = \partial_y H_y = (E_0/Z_0)\partial_y[\sin(\omega z/c)\sin(\omega t)] = 0.$$

b) At the interior facets of the mirrors, where $z = \pm d/2$, the E-field vanishes, simply because $\sin(\omega z/c) = \sin(\pm\,\omega d/2c) = \pm\sin(\omega n\lambda_0/2c) = \pm\sin(n\pi) = 0$. The tangential component E_x of the E-field is thus continuous, given that the E-field inside the PEC mirrors is also zero.

As for the perpendicular component of the E-field, Maxwell's first equation demands that any discontinuity in the perpendicular component of the D-field be accounted for by the presence of a surface charge-density σ_s at a mirror surface. In this problem, however, $D_\perp = \varepsilon_0 E_z$ is zero everywhere in the system, indicating that the mirror surfaces do not contain any free charges.

c) The H-field inside the PEC mirrors is zero, whereas immediately in front of the interior facet of each mirror, the H-field is given by $(-1)^n(E_0/Z_0)\cos(\omega t)\,\hat{\boldsymbol{y}}$. This is because, at $z = \pm d/2$, $\cos(\omega z/c) = \cos(\pm\omega d/2c) = \cos(n\pi) = (-1)^n$. Since the discontinuity of tangential H-field must be accompanied by a corresponding surface current-density $\boldsymbol{J}_s$, we conclude that $\boldsymbol{J}_s = (-1)^n(E_0/Z_0)\cos(\omega t)\,\hat{\boldsymbol{x}}$ at $z = d/2$, and $\boldsymbol{J}_s = (-1)^{n+1}(E_0/Z_0)\cos(\omega t)\,\hat{\boldsymbol{x}}$ at $z = -d/2$.

At the interior facet of each mirror, according to Maxwell's 4th equation, the perpendicular B-field component must be continuous. This is guaranteed in the present problem by the fact that $B_z = \mu_0 H_z$ is everywhere equal to zero.

Problem 3)

a) $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}} \;\rightarrow\; \varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho_{\text{free}} - \boldsymbol{\nabla}\cdot\boldsymbol{P}$. Conjugating both sides of the equation, we find $\varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}^* = \rho_{\text{free}}^* - \boldsymbol{\nabla}\cdot\boldsymbol{P}^*$, which indicates that the source distributions ρ_{free}^* and $\boldsymbol{P}^*$ together give rise to the E-field distribution $\boldsymbol{E}^*$.

b) Upon adding Maxwell's 1st equation to its complex conjugate, one arrives at

$$\varepsilon_0\boldsymbol{\nabla}\cdot(\boldsymbol{E}+\boldsymbol{E}^*) = (\rho_{\text{free}}+\rho_{\text{free}}^*) - \boldsymbol{\nabla}\cdot(\boldsymbol{P}+\boldsymbol{P}^*).$$

If we now divide both sides of the above equation by 2, we can conclude that the combined source distributions $½(\rho_{\text{free}}+\rho_{\text{free}}^*) = \text{Real}(\rho_{\text{free}}) = \rho'$ and $½(\boldsymbol{P}+\boldsymbol{P}^*) = \text{Real}(\boldsymbol{P}) = \boldsymbol{P}'$ give rise to the E-field distribution $½(\boldsymbol{E}+\boldsymbol{E}^*) = \text{Real}(\boldsymbol{E}) = \boldsymbol{E}'$.

c) Subtracting from Maxwell's 1st equation its complex conjugate, one arrives at

$$\varepsilon_0\boldsymbol{\nabla}\cdot(\boldsymbol{E}-\boldsymbol{E}^*) = (\rho_{\text{free}}-\rho_{\text{free}}^*) - \boldsymbol{\nabla}\cdot(\boldsymbol{P}-\boldsymbol{P}^*).$$

If we now divide both sides of the above equation by 2i, we can conclude that the combined source distributions $(\rho_{\text{free}}-\rho_{\text{free}}^*)/2\text{i} = \text{Imag}(\rho_{\text{free}}) = \rho''$ and $(\boldsymbol{P}-\boldsymbol{P}^*)/2\text{i} = \text{Imag}(\boldsymbol{P}) = \boldsymbol{P}''$ give rise to the E-field distribution $(\boldsymbol{E}-\boldsymbol{E}^*)/2\text{i} = \text{Imag}(\boldsymbol{E}) = \boldsymbol{E}''$.

d) The same procedures as above, applied to all four of Maxwell's equations, reveal that, if the (complex-valued) sources ρ_{free}, $\boldsymbol{J}_{\text{free}}$, $\boldsymbol{P}$ and $\boldsymbol{M}$ produce the fields $\boldsymbol{E}$ and $\boldsymbol{H}$, then the source distributions ρ_{free}^*, $\boldsymbol{J}_{\text{free}}^*$, $\boldsymbol{P}^*$ and $\boldsymbol{M}^*$ will produce the fields $\boldsymbol{E}^*$ and $\boldsymbol{H}^*$. By the same token, the real parts ρ_{free}', $\boldsymbol{J}_{\text{free}}'$, $\boldsymbol{P}'$ and $\boldsymbol{M}'$ of the sources give rise to $\boldsymbol{E}'$ and $\boldsymbol{H}'$. Similarly, the imaginary parts ρ_{free}'', $\boldsymbol{J}_{\text{free}}''$, $\boldsymbol{P}''$ and $\boldsymbol{M}''$ of the sources produce the fields $\boldsymbol{E}''$ and $\boldsymbol{H}''$.
