Opti 501 **1st Midterm Solutions** (9/27/2018) **Time: 75 minutes**

Problem 1) a) In free space, $\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P} = \varepsilon_0\boldsymbol{E}$, and $\boldsymbol{B} = \mu_0\boldsymbol{H} + \boldsymbol{M} = \mu_0\boldsymbol{H}$. Therefore,

i) $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E} = \varepsilon_0(\partial_x E_x + \partial_y E_y + \partial_z E_z) = 0.$

ii)
$$\boldsymbol{\nabla}\times\boldsymbol{H} = (\cancel{\partial_y H_z} - \cancel{\partial_z H_y})\hat{\boldsymbol{x}} + (\cancel{\partial_z H_x} - \partial_x H_z)\hat{\boldsymbol{y}} + (\partial_x H_y - \cancel{\partial_y H_x})\hat{\boldsymbol{z}}$$
$$= (E_0/Z_0)(\omega/c)\sin[(\omega/c)x-\omega t]\,\hat{\boldsymbol{y}} - (E_0/Z_0)(\omega/c)\cos[(\omega/c)x-\omega t]\,\hat{\boldsymbol{z}}$$
$$= \varepsilon_0 E_0\omega\sin[(\omega/c)x-\omega t]\,\hat{\boldsymbol{y}} - \varepsilon_0 E_0\omega\cos[(\omega/c)x-\omega t]\,\hat{\boldsymbol{z}}$$
$$\partial_t\boldsymbol{D} = \varepsilon_0\partial_t\boldsymbol{E} = \varepsilon_0 E_0\omega\sin[(\omega/c)x-\omega t]\,\hat{\boldsymbol{y}} - \varepsilon_0 E_0\omega\cos[(\omega/c)x-\omega t]\,\hat{\boldsymbol{z}}.$$

Clearly, $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \partial_t\boldsymbol{D}$ is satisfied.

iii)
$$\boldsymbol{\nabla}\times\boldsymbol{E} = (\cancel{\partial_y E_z} - \cancel{\partial_z E_y})\hat{\boldsymbol{x}} + (\cancel{\partial_z E_x} - \partial_x E_z)\hat{\boldsymbol{y}} + (\partial_x E_y - \cancel{\partial_y E_x})\hat{\boldsymbol{z}}$$
$$= -(\omega/c)E_0\cos[(\omega/c)x-\omega t]\,\hat{\boldsymbol{y}} - (\omega/c)E_0\sin[(\omega/c)x-\omega t]\,\hat{\boldsymbol{z}}.$$
$$\partial_t\boldsymbol{B} = \mu_0\partial_t\boldsymbol{H} = \mu_0(E_0/Z_0)\omega\sin[(\omega/c)x-\omega t]\,\hat{\boldsymbol{z}} + \mu_0(E_0/Z_0)\omega\cos[(\omega/c)x-\omega t]\,\hat{\boldsymbol{y}}$$
$$= (\omega/c)E_0\sin[(\omega/c)x-\omega t]\,\hat{\boldsymbol{z}} + (\omega/c)E_0\cos[(\omega/c)x-\omega t]\,\hat{\boldsymbol{y}}.$$

It is seen that Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B}$ is satisfied.

iv) $\boldsymbol{\nabla}\cdot\boldsymbol{B} = \mu_0(\partial_x H_x + \partial_y H_y + \partial_z H_z) = 0.$

b) $\mathbb{E}(\boldsymbol{r},t) = \tfrac{1}{2}\varepsilon_0\boldsymbol{E}\cdot\boldsymbol{E} = \tfrac{1}{2}\varepsilon_0 E_0^2\{\cos^2[(\omega/c)x-\omega t] + \sin^2[(\omega/c)x-\omega t]\} = \tfrac{1}{2}\varepsilon_0 E_0^2.$

$\mathbb{H}(\boldsymbol{r},t) = \tfrac{1}{2}\mu_0\boldsymbol{H}\cdot\boldsymbol{H} = \tfrac{1}{2}\mu_0(E_0/Z_0)^2\{\cos^2[(\omega/c)x-\omega t] + \sin^2[(\omega/c)x-\omega t]\} = \tfrac{1}{2}\varepsilon_0 E_0^2.$

$$\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}\times\boldsymbol{H} = (E_y H_z - E_z H_y)\hat{\boldsymbol{x}}$$
$$= (E_0^2/Z_0)\{\cos^2[(\omega/c)x-\omega t] + \sin^2[(\omega/c)x-\omega t]\}\hat{\boldsymbol{x}} = (E_0^2/Z_0)\hat{\boldsymbol{x}}.$$

c) $$\boldsymbol{\nabla}\cdot\boldsymbol{S} + \partial_t(\mathbb{E}+\mathbb{H}) = \partial_x(E_0^2/Z_0) + \partial_t(\varepsilon_0 E_0^2) = 0.$$

Problem 2) The surface-charge-density is σ_{s0}, and the surface-current-density, which is the product of the surface charge-density and the local velocity at the cylinder surface, is $\boldsymbol{J}_{s0} = R\omega\sigma_{s0}\hat{\boldsymbol{\varphi}}$. In this problem $\boldsymbol{P}(\boldsymbol{r},t) = 0$ and $\boldsymbol{M}(\boldsymbol{r},t) = 0$. The discontinuity in the perpendicular component of $\boldsymbol{D}$ must, therefore, be equal to σ_{s0} (because $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}}$), that is,

$$D_\perp(\rho = R^+,\varphi,z,t) - D_\perp(\rho = R^-,\varphi,z,t) = \sigma_{s0},$$

or, equivalently,

$$\varepsilon_0 E_\rho(R^+,\varphi,z,t) - \varepsilon_0 E_\rho(R^-,\varphi,z,t) = \sigma_{s0}.$$

The perpendicular component of $\boldsymbol{B}$ must be continuous (because $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$). Consequently,

$$\mu_0 H_\rho(R^+,\varphi,z,t) = \mu_0 H_\rho(R^-,\varphi,z,t).$$

Both tangential components of $\boldsymbol{E}$ must be continuous (because $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B}$). Therefore,

$$E_\varphi(R^+,\varphi,z,t) = E_\varphi(R^-,\varphi,z,t).$$

$$E_z(R^+,\varphi,z,t) = E_z(R^-,\varphi,z,t).$$

The tangential component of $\boldsymbol{H}$ that is parallel to the surface current is also continuous (because $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \partial_t\boldsymbol{D}$), that is,

$$H_\varphi(R^+,\varphi,z,t) = H_\varphi(R^-,\varphi,z,t).$$

However, the tangential component of $\boldsymbol{H}$ that is orthogonal to the surface current has a discontinuity equal to the surface current density, that is,

$$H_z(R^+,\varphi,z,t) - H_z(R^-,\varphi,z,t) = -J_{s,\varphi}(\rho = R,\varphi,z,t) = -R\omega\sigma_{s0}.$$

In the preceding equation, the rotation of the hollow cylinder, seen from above, is assumed to be counterclockwise. The right-hand rule then dictates the sign of the discontinuity of H_z.

Problem 3) The surface-charge-density is σ_{s0}, and the surface-current-density, which is the product of the surface charge-density and the local velocity at the sphere's surface, is $\boldsymbol{J}_{s0} = (R\sin\theta)\omega\sigma_{s0}\hat{\boldsymbol{\varphi}}$. In this problem $\boldsymbol{P}(\boldsymbol{r},t) = 0$ and $\boldsymbol{M}(\boldsymbol{r},t) = 0$. The discontinuity in the perpendicular component of $\boldsymbol{D}$ must, therefore, be equal to σ_{s0} (because $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}}$), that is,

$$D_\perp(r = R^+,\theta,\varphi,t) - D_\perp(r = R^-,\theta,\varphi,t) = \sigma_{s0},$$

or, equivalently,

$$\varepsilon_0 E_r(R^+,\theta,\varphi,t) - \varepsilon_0 E_r(R^-,\theta,\varphi,t) = \sigma_{s0}.$$

The perpendicular component of $\boldsymbol{B}$ must be continuous (because $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$). Consequently,

$$\mu_0 H_r(R^+,\theta,\varphi,t) = \mu_0 H_r(R^-,\theta,\varphi,t).$$

Both tangential components of $\boldsymbol{E}$ must be continuous (because $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B}$). Therefore,

$$E_\theta(R^+,\theta,\varphi,t) = E_\theta(R^-,\theta,\varphi,t).$$

$$E_\varphi(R^+,\theta,\varphi,t) = E_\varphi(R^-,\theta,\varphi,t).$$

The tangential component of $\boldsymbol{H}$ that is parallel to the surface current is also continuous (because $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \partial_t\boldsymbol{D}$), that is,

$$H_\varphi(R^+,\theta,\varphi,t) = H_\varphi(R^-,\theta,\varphi,t).$$

However, the tangential component of $\boldsymbol{H}$ that is orthogonal to the surface current has a discontinuity equal to the surface current density, that is,

$$H_\theta(R^+,\theta,\varphi,t) - H_\theta(R^-,\theta,\varphi,t) = J_{s,\varphi}(\rho = R,\theta,\varphi,t) = (R\sin\theta)\omega\sigma_{s0}.$$

In the preceding equation, the rotation of the spherical shell, seen from above, is assumed to be counterclockwise. The right-hand rule then dictates the sign of the discontinuity of H_θ.

Problem 4) a) Consider a closed surface inside the metallic host that encloses the cavity. Since the E-field everywhere on this closed surface must vanish, the integral of the E-field over the closed surface will be zero. According to Maxwell's 1^{st} equation, the total charge inside the closed surface must vanish, that is, the overall charge induced on the walls of the cavity must be equal in magnitude and opposite in sign to the charge q that has been placed inside the cavity.

b) The above argument indicates that, when the charge q is removed from the cavity, the overall induced charge on the walls of the cavity must go to zero. However, the argument based on Maxwell's 1^{st} equation does *not* say anything at all about the *distribution* of charge-density over the walls. In other words, it is still possible to have equal amounts of positive and negative charge at different locations on the wall.

Now, the 3^{rd} of Maxwell's equations, $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$, must also be satisfied in this electrostatic problem. Since the E-field lines originate on positive charges and terminate on negative charges, if we define a closed loop that starts on a positive charge at some point on the cavity wall, follows an E-field line until it reaches a negative charge elsewhere on the cavity wall, then closes the path by going through the metallic host (where the E-field is zero), then the integral of the E-field around such a loop will *not* vanish. Given that the satisfaction of Maxwell's 3^{rd} equation is mandatory, we conclude that separated positive and negative charges cannot exist anywhere on the cavity walls, once the charge q is taken out of the cavity.
