Opti 501 **1st Midterm Solutions** (9/28/2017) **Time: 75 minutes**

Problem 1)

$$\boldsymbol{E}(\boldsymbol{r},t) = \exp(\gamma t - \boldsymbol{\beta}\cdot\boldsymbol{r})\,[\boldsymbol{A}\cos(\boldsymbol{\alpha}\cdot\boldsymbol{r} - \omega_0 t + \varphi_A) - \boldsymbol{B}\sin(\boldsymbol{\alpha}\cdot\boldsymbol{r} - \omega_0 t + \varphi_B)]$$

$$= \exp(\gamma t - \boldsymbol{\beta}\cdot\boldsymbol{r})\,\{\text{Real}\{\boldsymbol{A}\exp[\mathrm{i}(\boldsymbol{\alpha}\cdot\boldsymbol{r} - \omega_0 t + \varphi_A)]\} - \text{Imag}\{\boldsymbol{B}\exp[\mathrm{i}(\boldsymbol{\alpha}\cdot\boldsymbol{r} - \omega_0 t + \varphi_B)]\}\}$$

$$= \exp(\gamma t - \boldsymbol{\beta}\cdot\boldsymbol{r})$$
$$\times\{\text{Real}\{\boldsymbol{A}\exp(\mathrm{i}\varphi_A)\exp[\mathrm{i}(\boldsymbol{\alpha}\cdot\boldsymbol{r} - \omega_0 t)]\} - \text{Imag}\{\boldsymbol{B}\exp(\mathrm{i}\varphi_B)\exp[\mathrm{i}(\boldsymbol{\alpha}\cdot\boldsymbol{r} - \omega_0 t)]\}\}$$

$$= \exp(\gamma t - \boldsymbol{\beta}\cdot\boldsymbol{r})$$
$$\times\{\text{Real}\{\boldsymbol{A}\exp(\mathrm{i}\varphi_A)\exp[\mathrm{i}(\boldsymbol{\alpha}\cdot\boldsymbol{r} - \omega_0 t)]\} + \text{Real}\{\mathrm{i}\boldsymbol{B}\exp(\mathrm{i}\varphi_B)\exp[\mathrm{i}(\boldsymbol{\alpha}\cdot\boldsymbol{r} - \omega_0 t)]\}\}$$

$$= \exp(\gamma t - \boldsymbol{\beta}\cdot\boldsymbol{r})\,\text{Real}\{[\boldsymbol{A}\exp(\mathrm{i}\varphi_A) + \mathrm{i}\boldsymbol{B}\exp(\mathrm{i}\varphi_B)]\exp[\mathrm{i}(\boldsymbol{\alpha}\cdot\boldsymbol{r} - \omega_0 t)]\}$$

$$= \text{Real}\{[\boldsymbol{A}\exp(\mathrm{i}\varphi_A) + \mathrm{i}\boldsymbol{B}\exp(\mathrm{i}\varphi_B)]\exp\{\mathrm{i}[(\boldsymbol{\alpha} + \mathrm{i}\boldsymbol{\beta})\cdot\boldsymbol{r} - (\omega_0 + \mathrm{i}\gamma)t]\}\}.$$

Comparison with the complex-valued E-field reveals that $\boldsymbol{k} = \boldsymbol{\alpha} + \mathrm{i}\boldsymbol{\beta}$, $\omega = \omega_0 + \mathrm{i}\gamma$, and $\boldsymbol{E}_0 = \boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0'' = \boldsymbol{A}\exp(\mathrm{i}\varphi_A) + \mathrm{i}\boldsymbol{B}\exp(\mathrm{i}\varphi_B) = (\boldsymbol{A}\cos\varphi_A - \boldsymbol{B}\sin\varphi_B) + \mathrm{i}(\boldsymbol{A}\sin\varphi_A + \boldsymbol{B}\cos\varphi_B)$. If need be, one may also solve the expressions of $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ for arbitrary values of φ_A and φ_B to obtain

$$\boldsymbol{A} = \frac{(\cos\varphi_B)\boldsymbol{E}_0' + (\sin\varphi_B)\boldsymbol{E}_0''}{\cos(\varphi_A - \varphi_B)}; \qquad \boldsymbol{B} = \frac{-(\sin\varphi_A)\boldsymbol{E}_0' + (\cos\varphi_A)\boldsymbol{E}_0''}{\cos(\varphi_A - \varphi_B)}.$$

Problem 2)

a) $$\boldsymbol{\nabla}\cdot\boldsymbol{B} = \frac{\partial(\rho B_\rho)}{\rho\partial\rho} + \frac{\partial B_z}{\partial z}$$
$$= B_0\{2(z/z_0^2)[1 - (\rho/\rho_0)^2] + 2(z/z_0^2)[(\rho/\rho_0)^2 - 1]\}\exp[-(\rho/\rho_0)^2 - (z/z_0)^2] = 0.$$

b) $$\boldsymbol{J}_{\text{free}}(\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r}) = \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{B}(\boldsymbol{r}) = \mu_0^{-1}\left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial\rho}\right)\hat{\boldsymbol{\varphi}}$$
$$= \mu_0^{-1}B_0(\rho/\rho_0^2)\{(\rho_0/z_0)^2[1 - 2(z/z_0)^2] - 2(\rho/\rho_0)^2 + 4\}$$
$$\times\exp[-(\rho/\rho_0)^2 - (z/z_0)^2]\,\hat{\boldsymbol{\varphi}}.$$

c) As expected, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}}(\boldsymbol{r}) = 0$, which is consistent with the charge-current continuity equation for a static system where $\partial\rho_{\text{free}}/\partial t = 0$. The divergence of $\boldsymbol{J}_{\text{free}}$ may, of course, be evaluated directly from the above expression. However, since $\boldsymbol{J}_{\text{free}} = \boldsymbol{\nabla}\times\boldsymbol{H}$ and the divergence of curl is always zero, we readily conclude that $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{free}} = 0$.

Problem 3)

a) $$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{r}\left[\frac{\partial(rE_\theta)}{\partial r} - \frac{\partial E_r}{\partial\theta}\right]\hat{\boldsymbol{\varphi}} = \frac{E_0}{r}\left\{-r_0\cos\theta\,\frac{\partial\exp[-(r/r_0)^2]}{\partial r} - 2(r/r_0)\exp[-(r/r_0)^2]\frac{\partial\sin\theta}{\partial\theta}\right\}\hat{\boldsymbol{\varphi}}$$
$$= (E_0/r)[2(r/r_0)\cos\theta - 2(r/r_0)\cos\theta]\exp[-(r/r_0)^2]\,\hat{\boldsymbol{\varphi}} = 0.$$

b) $$\rho_{\text{free}}(\boldsymbol{r}) = \varepsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r}) = \varepsilon_0\left[\frac{\partial(r^2E_r)}{r^2\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\theta)}{\partial\theta}\right]$$

$$= \varepsilon_0 E_0 \left\{\frac{\partial}{r^2 \partial r} 2(r^3/r_0) \exp[-(r/r_0)^2] \sin\theta - \frac{(r_0/r)\exp[-(r/r_0)^2]}{r\sin\theta}\frac{\partial(\sin\theta\cos\theta)}{\partial\theta}\right\}$$

$$= (\varepsilon_0 E_0/r_0)\left\{2[3 - 2(r/r_0)^2]\sin\theta - \frac{(r_0/r)^2\cos(2\theta)}{\sin\theta}\right\}\exp[-(r/r_0)^2].$$

c) $$Q = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi} 2\pi r^2 \sin\theta\, \rho_{\text{free}}(r,\theta)\mathrm{d}r\mathrm{d}\theta$$

$$= 2\pi r_0 \varepsilon_0 E_0 \int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\{2(r/r_0)^2[3 - 2(r/r_0)^2]\sin^2\theta - \cos(2\theta)\}\exp[-(r/r_0)^2]\,\mathrm{d}r\mathrm{d}\theta$$

$$= 2\pi^2 r_0 \varepsilon_0 E_0 \int_0^{\infty}(r/r_0)^2[3 - 2(r/r_0)^2]\exp[-(r/r_0)^2]\,\mathrm{d}r$$

$$= 2\pi^2 r_0^2 \varepsilon_0 E_0 \int_0^{\infty}(3x^2 - 2x^4)\exp(-x^2)\,\mathrm{d}x = 2\pi^2 r_0^2 \varepsilon_0 E_0\left(\frac{3\sqrt{\pi}}{4} - \frac{6\sqrt{\pi}}{8}\right) = 0.$$

The above result should be expected because, when $r \to \infty$, $\boldsymbol{E}(\boldsymbol{r}) \to 0$ in such a way that the integral of $\varepsilon_0 \boldsymbol{E}(\boldsymbol{r})$ over the surface of an infinitely large sphere approaches zero. Consequently, in accordance with Maxwell's 1st equation, the total charge Q inside the (infinitely large) sphere must vanish.

Problem 4)

a) In general, $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$ and $\boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{M}$. Maxwell's equations in differential form are written as follows:

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}}, \tag{1}$$

$$\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \partial\boldsymbol{D}/\partial t, \tag{2}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\,\partial\boldsymbol{B}/\partial t, \tag{3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0. \tag{4}$$

b) Upon elimination of $\boldsymbol{E}$ and $\boldsymbol{H}$, Eqs.(1) and (4) remain intact, whereas Eqs.(2) and (3) become

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0(\boldsymbol{J}_{\text{free}} + \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}) + \mu_0\,\partial\boldsymbol{D}/\partial t, \tag{2$'$}$$

$$\boldsymbol{\nabla}\times\boldsymbol{D} = \varepsilon_0(\varepsilon_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{P} - \partial\boldsymbol{B}/\partial t). \tag{3$'$}$$

c) From Eqs.(1) and (4) we infer that, in the present formulation, the total electric charge-density is $\rho_{\text{total}}^{(e)} = \rho_{\text{free}}$, while the total magnetic charge-density is $\rho_{\text{total}}^{(m)} = 0$. The total electric current-density is seen from Eq.(2′) to be $\boldsymbol{J}_{\text{total}}^{(e)} = \boldsymbol{J}_{\text{free}} + \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}$. Considering the charge-current continuity equation, $\boldsymbol{\nabla}\cdot\boldsymbol{J} + (\partial\rho/\partial t) = 0$, and that, according to a well-known vector identity, $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{M}) = 0$, it is seen that no electric charge-density is associated with $\boldsymbol{J}_{\text{bound}}^{(e)} = \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}$.

Similarly, according to Eq.(3′), the magnetic current-density is $\boldsymbol{J}_{\text{total}}^{(m)} = \boldsymbol{J}_{\text{bound}}^{(m)} = \varepsilon_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}$. As before, the charge-current continuity equation, $\boldsymbol{\nabla}\cdot\boldsymbol{J} + (\partial\rho/\partial t) = 0$, in conjunction with the vector identity $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{P}) = 0$ implies that $\rho_{\text{total}}^{(m)} = 0$, in agreement with Eq.(4). Note that the units of $\varepsilon_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}$ are the same as those of $\partial\boldsymbol{B}/\partial t$, namely, weber/(m$^2\cdot$sec), which is consistent with the designation of $\varepsilon_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}$ as the bound magnetic current-density $\boldsymbol{J}_{\text{bound}}^{(m)}$.

d) Upon dot-multiplying Eq.(2′) by $\boldsymbol{D}$ and Eq.(3′) by $\boldsymbol{B}$ we will have

$$\boldsymbol{D}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) = \mu_0\boldsymbol{D}\cdot\boldsymbol{J}_{\text{free}} + \boldsymbol{D}\cdot(\boldsymbol{\nabla}\times\boldsymbol{M}) + \mu_0\boldsymbol{D}\cdot(\partial\boldsymbol{D}/\partial t), \tag{5}$$

$$\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{D}) = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{P}) - \varepsilon_0\boldsymbol{B}\cdot(\partial\boldsymbol{B}/\partial t). \tag{6}$$

Subtracting Eq.(6) from Eq.(5) and using the vector identity $\boldsymbol{D}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) - \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{D}) = \boldsymbol{\nabla}\cdot(\boldsymbol{B}\times\boldsymbol{D})$ now yields

$$\boldsymbol{\nabla}\cdot(\boldsymbol{B}\times\boldsymbol{D}) = \frac{\partial}{\partial t}(½\mu_0\boldsymbol{D}\cdot\boldsymbol{D} + ½\varepsilon_0\boldsymbol{B}\cdot\boldsymbol{B}) + \mu_0\boldsymbol{D}\cdot\boldsymbol{J}_{\text{free}} + \boldsymbol{D}\cdot(\boldsymbol{\nabla}\times\boldsymbol{M}) - \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{P}). \tag{7}$$

We multiply both sides of the above equation by $c^2 = 1/(\mu_0\varepsilon_0)$, then define the alternative Poynting vector $\boldsymbol{S}(\boldsymbol{r},t) = c^2\boldsymbol{D}(\boldsymbol{r},t)\times\boldsymbol{B}(\boldsymbol{r},t)$, which has the units of $Joule/(m^2\cdot\text{sec})$, to arrive at

$$\boldsymbol{\nabla}\cdot\boldsymbol{S} + \frac{\partial}{\partial t}(½\varepsilon_0^{-1}\boldsymbol{D}\cdot\boldsymbol{D} + ½\mu_0^{-1}\boldsymbol{B}\cdot\boldsymbol{B})$$
$$+(\boldsymbol{D}/\varepsilon_0)\cdot(\boldsymbol{J}_{\text{free}} + \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}) - (\boldsymbol{B}/\mu_0)\cdot(\varepsilon_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}) = 0. \tag{8}$$

It is thus seen in the proposed formulation that the D-field energy-density is $½\varepsilon_0^{-1}\boldsymbol{D}\cdot\boldsymbol{D}$, while that of the B-field is $½\mu_0^{-1}\boldsymbol{B}\cdot\boldsymbol{B}$. The D-field exchanges energy with the electric current-density $\boldsymbol{J}_{\text{total}}^{(e)}$ at the rate of $(\boldsymbol{D}/\varepsilon_0)\cdot\boldsymbol{J}_{\text{total}}^{(e)}$, whereas the B-field exchanges energy with the magnetic current-density $\boldsymbol{J}_{\text{total}}^{(m)}$ at the rate of $-(\boldsymbol{B}/\mu_0)\cdot\boldsymbol{J}_{\text{total}}^{(m)}$.