Opti 501 **1^{st} Midterm Solutions** (10/6/2016) **Time: 75 minutes**

Solution to Problem 1) a) Using Gauss' law in conjunction with spherical surfaces of radius ρ, we find, since the E-field inside the metallic shell of the inner sphere must vanish, that the total charge on the interior surface of the inner sphere must be zero. Therefore, $\sigma_{12} = 0$. The charge content of the inner sphere is thus distributed entirely on its outer surface, namely, $\sigma_{11} = \sigma_1$.

As for the outer sphere, we place the Gaussian surface inside its metallic shell. Since the E-field inside the metal must be zero, the total charge inside the Gaussian sphere must vanish, that is, $4\pi R_1^2\sigma_1 + 4\pi R_2^2\sigma_{22} = 0$. This yields $\sigma_{22} = -(R_1/R_2)^2\sigma_1$. The remaining charge will then appear on the outer surface of the outer sphere, that is, $\sigma_{21} = \sigma_2 - \sigma_{22}$.

b) Inside the small sphere the E-field is zero. The total charge on the inner shell is $Q = 4\pi R_1^2\sigma_1$. Therefore, in the region between the two spheres,

$$\boldsymbol{E}_1(\rho) = (Q/4\pi\varepsilon_o\rho^2)\hat{\boldsymbol{\rho}} = (\sigma_1/\varepsilon_o)\,(R_1/\rho)^2\hat{\boldsymbol{\rho}}; \qquad R_1 < \rho < R_2.$$

Outside the large sphere, the fields of the two spheres are superimposed, that is,

$$\boldsymbol{E}_2(\rho) = [(\sigma_1/\varepsilon_o)\,(R_1/\rho)^2 + (\sigma_2/\varepsilon_o)\,(R_2/\rho)^2]\hat{\boldsymbol{\rho}}; \qquad \rho > R_2.$$

c) The potential difference (i.e., voltage) between the spheres is given by the integral of $\boldsymbol{E}_1(\rho)$ along the radial direction from R_1 to R_2, that is,

$$V_{12} = \int_{R_1}^{R_2} E_{1\rho}(\rho)\mathrm{d}\rho = \int_{R_1}^{R_2}(\sigma_1/\varepsilon_o)\,(R_1/\rho)^2\mathrm{d}\rho = -(\sigma_1 R_1^2/\varepsilon_o)\,[(1/R_2) - (1/R_1)]$$

$$= (\sigma_1/\varepsilon_o)(R_1/R_2)(R_2 - R_1).$$

Therefore,

$$C = Q/V = \frac{4\pi R_1^2\sigma_1}{(\sigma_1/\varepsilon_o)(R_1/R_2)(R_2-R_1)} = 4\pi\varepsilon_o R_1 R_2/(R_2 - R_1).$$

Solution to Problem 2) a) The surface-current-density $\boldsymbol{J}_s$ is the product of ordinary current-density $\boldsymbol{J}_{\text{free}}$ and the *very small* thickness τ of the cylindrical shell, that is, $\boldsymbol{J}_s = \boldsymbol{J}_{\text{free}}\tau$. Considering that the units of $\boldsymbol{J}_{\text{free}}$ are ampere/m^2, we conclude that $\boldsymbol{J}_s$ has units of ampere/m.

b) In a cylindrical coordinate system the H-field has three components, namely H_ρ, H_φ, and H_z. Due to the symmetry of the setup, these field components must be independent of the φ and z coordinates. This is because the current-carrying cylinder would look exactly the same if the ρ and z coordinates of the observation point were fixed while its φ coordinate varied. Similarly, the system would look the same if the ρ and φ coordinates of the observation point were fixed while its z coordinate varied. The H-field components, therefore, can only be functions of the ρ coordinate.

c) In conjunction with Maxwell's 4^{th} equation, $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, we use a cylindrical volume of radius ρ and length L, centered on the z-axis, to show that $H_\rho(\rho) = 0$. Clearly, $H_\varphi(\rho)$ does not contribute to the surface integral of $\boldsymbol{B} = \mu_o\boldsymbol{H}$ over the cylinder. Also, contributions from $H_z(\rho)$ to the top and bottom facets of the cylinder cancel each other out. The contribution of $H_\rho(\rho)$ to the surface integral is $2\pi\rho L H_\rho(\rho)$, but the overall surface integral must be zero and, therefore, $H_\rho(\rho) = 0$.

Next, we use a circular loop of radius ρ in the xy-plane in conjunction with Maxwell's 2^{nd} equation, $\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\text{free}}$, to demonstrate that $H_\varphi(\rho) = 0$. The line integral of H_φ around the loop is $2\pi\rho H_\varphi(\rho)$. However, no current crosses the loop and, therefore, $H_\varphi(\rho) = 0$.

Finally, we use rectangular loops in the ρz-plane, again in conjunction with Maxwell's 2^{nd} equation, to obtain information about $H_z(\rho)$. If the rectangular loop is placed entirely outside the cylinder we find that no current crosses the loop and that, therefore, H_z outside the cylinder is uniform. Similarly, placing the rectangular loop inside the cylinder shows that H_z inside the cylinder is uniform as well. However, if the $L \times W$ rectangle has one leg inside and the opposite leg outside the cylinder, the integral of H_z around the loop will be $[H_z^{(\text{inside})} - H_z^{(\text{outside})}]L$, while the current that crosses the loop will be $J_{s0}L$. Consequently, $H_z^{(\text{inside})} - H_z^{(\text{outside})} = J_{s0}$.

Given that a uniform H_z field residing in the entire space cannot, in any way, be related via Maxwell's equations to the solenoidal current $J_{s0}\hat{\boldsymbol{\varphi}}$, we conclude that $H_z^{(\text{outside})} = 0$. Therefore,

$$\boldsymbol{H}(\rho, \varphi, z) = \begin{cases} J_{s0}\hat{\boldsymbol{z}}; & 0 \le \rho < R, \\ 0; & \qquad \rho > R. \end{cases}$$

Solution to Problem 3) They are all correct. Alice writes Maxwell's equations as follows:

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_{\text{free}} - \boldsymbol{\nabla} \cdot \boldsymbol{P}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J}_{\text{free}} + \partial \boldsymbol{P}/\partial t + \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{M}\right) + \mu_0 \varepsilon_0\, \partial \boldsymbol{E}/\partial t$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\, \partial \boldsymbol{B}/\partial t$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

In her approach, polarization produces electric charges and currents, while magnetization produces electric currents only. Alice then bundles all charge densities together, and all current densities together, and proceeds to use her equations to compute the resulting $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

In contrast, Brian writes Maxwell's equations as follows:

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_{\text{free}}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \partial \boldsymbol{D}/\partial t$$

$$\boldsymbol{\nabla} \times \boldsymbol{D} = -\varepsilon_0(\partial \boldsymbol{M}/\partial t - \varepsilon_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{P}) - \mu_0 \varepsilon_0 \partial \boldsymbol{H}/\partial t$$

$$\mu_0 \boldsymbol{\nabla} \cdot \boldsymbol{H} = -\boldsymbol{\nabla} \cdot \boldsymbol{M}$$

In this approach, magnetization produces magnetic charges and currents, while polarization produces magnetic currents only. Brian then bundles the two magnetic current-densities together, while treating electric and magnetic charge-densities separately. He proceeds to use his equations to compute the resulting $\boldsymbol{D}$ and $\boldsymbol{H}$ fields.

Carol writes Maxwell's equations as follows:

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_{\text{free}} - \boldsymbol{\nabla} \cdot \boldsymbol{P}$$

$$\boldsymbol{\nabla}\times\boldsymbol{H} = (\boldsymbol{J}_{\text{free}} + \partial\boldsymbol{P}/\partial t) + \varepsilon_o\,\partial\boldsymbol{E}/\partial t$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\,\partial\boldsymbol{M}/\partial t - \mu_o\partial\boldsymbol{H}/\partial t$$

$$\mu_o\boldsymbol{\nabla}\cdot\boldsymbol{H} = -\boldsymbol{\nabla}\cdot\boldsymbol{M}$$

In her approach, polarization produces electric charges and currents, while magnetization produces magnetic charges and currents. Carol treats the two types of charge-density separately, and also the two types of current-density separately. She then proceeds to use her version of Maxwell's equations to compute the resulting $\boldsymbol{E}$ and $\boldsymbol{H}$ fields.

David writes Maxwell's equations as follows:

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}},$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_o\,(\boldsymbol{J}_{\text{free}} + \mu_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}) + \mu_o\,\partial\boldsymbol{D}/\partial t,$$

$$\boldsymbol{\nabla}\times\boldsymbol{D} = -\varepsilon_o(-\varepsilon_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}) - \varepsilon_o\partial\boldsymbol{B}/\partial t,$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0.$$

In David's approach, polarization produces magnetic currents, while magnetization produces electric currents. David then treats the two types of current-density separately, and proceeds to use his version of Maxwell's equations to compute the resulting $\boldsymbol{D}$ and $\boldsymbol{B}$ fields.

Solution to Problem 4) a) For a relativistic treatment of the problem, define $\beta_{0,1} = V_{0,1}/c$ and $\gamma_{0,1} = 1/\sqrt{1-(V_{0,1}/c)^2}$. The conservation laws of energy and linear momentum may then be written as follows:

$$\mathcal{E}_0 + \gamma_0 Mc^2 = \mathcal{E}_1 + \gamma_1 Mc^2, \tag{1a}$$

$$(\mathcal{E}_0/c) + \gamma_0 MV_0 = -(\mathcal{E}_1/c) + \gamma_1 MV_1. \tag{1b}$$

Note that $\mathcal{E}_0$ and M can have arbitrary (positive) values, and that V_0 may be positive, zero, or negative, provided that $|V_0| < c$. Defining $\alpha_{0,1} = \mathcal{E}_{0,1}/Mc^2$, the above equations can be written in somewhat simplified form as

$$\alpha_0 + \gamma_0 = \alpha_1 + \gamma_1, \tag{2a}$$

$$\alpha_0 + \gamma_0\beta_0 = -\alpha_1 + \gamma_1\beta_1. \tag{2b}$$

b) In the non-relativistic approximation, we have

$$\mathcal{E}_0 + ½MV_0^2 = \mathcal{E}_1 + ½MV_1^2, \tag{3a}$$

$$(\mathcal{E}_0/c) + MV_0 = -(\mathcal{E}_1/c) + MV_1. \tag{3b}$$

After normalization, Eqs.(3a) and (3b) become

$$\alpha_0 + ½\beta_0^2 = \alpha_1 + ½\beta_1^2, \tag{4a}$$

$$\alpha_0 + \beta_0 = -\alpha_1 + \beta_1. \tag{4b}$$

c) Adding Eq.(4a) to Eq.(4b) and rearranging the terms, we find

$$\beta_1^2 + 2\beta_1 - (4\alpha_0 + 2\beta_0 + \beta_0^2) = 0. \tag{5}$$

Considering that $|\beta_1| = |V_1|/c$ must be less than 1.0, only one of the two solutions of the above quadratic equation in β_1 will be acceptable, that is,

$$\beta_1 = \sqrt{1 + 4\alpha_0 + 2\beta_0 + \beta_0^2} - 1. \tag{6}$$

Substitution into Eq.(4b) then yields

$$\mathcal{E}_1 = Mc^2(\beta_1 - \beta_0) - \mathcal{E}_0. \tag{7}$$

d) Given that, in the non-relativistic regime, $\alpha_0 \ll 1$, $\beta_0 \ll 1$, and $\beta_1 \ll 1$, we can approximate β_1 of Eq.(6) by invoking the Taylor series expansion $\sqrt{1+\varepsilon} = 1 + ½\varepsilon - ⅛\varepsilon^2 + \cdots$, as follows:

$$\begin{aligned}\beta_1 &= (2\alpha_0 + \beta_0 + ½\beta_0^2) - ⅛(4\alpha_0 + 2\beta_0 + \beta_0^2)^2 + \cdots \\ &= 2\alpha_0 + \beta_0 - 2\alpha_0^2 - 2\alpha_0\beta_0 - (\alpha_0 + ½\beta_0 + ⅛\beta_0^2)\beta_0^2 + \cdots \\ &\cong \beta_0 + 2\alpha_0(1 - \alpha_0 - \beta_0).\end{aligned} \tag{8}$$

ignore high-order terms

Thus, to a good approximation, Eq.(8) provides an expression for the final velocity V_1 of the mirror in terms of its initial velocity V_0, the energy $\mathcal{E}_0$ of the light bullet, and the mass M of the mirror. Substitution from Eq.(8) into Eq.(7) yields the final energy of the light pulse, as follows:

$$\mathcal{E}_1 \cong 2\mathcal{E}_0(1 - \alpha_0 - \beta_0) - \mathcal{E}_0 \quad \rightarrow \quad \mathcal{E}_1/\mathcal{E}_0 \cong 1 - 2(\mathcal{E}_0/Mc^2) - 2(V_0/c). \tag{9}$$

In the quantum picture of light, the incident pulse contains N photons of (angular) frequency ω_0 and energy $\hbar\omega_0$, so that $\mathcal{E}_0 = N\hbar\omega_0$. Upon encountering the mirror, all N photons are reflected, with their frequencies Doppler-shifted to ω_1, so that $\mathcal{E}_1 = N\hbar\omega_1$. Thus, the Doppler shift of the optical frequency upon perfect reflection from a moving (or stationary) mirror fully accounts for the change of the pulse energy from $\mathcal{E}_0$ to $\mathcal{E}_1$. If the term $2\mathcal{E}_0/(Mc^2)$ in Eq.(9) happens to be negligible, then the Doppler shift will be $\Delta\omega = \omega_1 - \omega_0 \cong -2(V_0/c)\omega_0$. Note that V_0 could be positive or negative, and that, therefore, the Doppler shift could decrease or increase the frequency of the light pulse upon reflection. For a stationary mirror (i.e., $V_0 = 0$), the kinetic energy acquired by the mirror after reflection of the light pulse will be $2\mathcal{E}_0^2/(Mc^2)$. The more massive the stationary mirror, the smaller will be the fraction of the energy of the pulse that is converted to the mirror's kinetic energy. Also, the greater the energy of the incident light pulse, the greater will be the fraction of its energy converted to the kinetic energy of the mirror.

Digression: In the relativistic treatment of part (a), adding Eq.(2a) to Eq.(2b) yields

$$2\alpha_0 + \gamma_0(1 + \beta_0) = \gamma_1(1 + \beta_1) = \sqrt{(1 + \beta_1)/(1 - \beta_1)}. \tag{10}$$

The above equation may now be solved to yield β_1, as follows:

$$\frac{1+\beta_1}{1-\beta_1} = [2\alpha_0 + \gamma_0(1 + \beta_0)]^2 \quad \rightarrow \quad \beta_1 = \frac{[2\alpha_0+\gamma_0(1+\beta_0)]^2 - 1}{[2\alpha_0+\gamma_0(1+\beta_0)]^2 + 1}. \tag{11}$$

Having found β_1, we can now derive an expression for γ_1, namely,

$$1 - \beta_1^2 = \frac{\{[2\alpha_0+\gamma_0(1+\beta_0)]^2+1\}^2 - \{[2\alpha_0+\gamma_0(1+\beta_0)]^2-1\}^2}{\{[2\alpha_0+\gamma_0(1+\beta_0)]^2+1\}^2} \quad \rightarrow \quad \sqrt{1-\beta_1^2} = \frac{2[2\alpha_0+\gamma_0(1+\beta_0)]}{[2\alpha_0+\gamma_0(1+\beta_0)]^2+1}$$

$$\rightarrow \quad \gamma_1 = 1/\sqrt{1-\beta_1^2} = \frac{[2\alpha_0+\gamma_0(1+\beta_0)]^2+1}{2[2\alpha_0+\gamma_0(1+\beta_0)]}. \tag{12}$$

Substitution for γ_1 into Eq.(2a) now yields a solution for α_1, as follows:

$$\alpha_1 = \alpha_0 + \gamma_0 - \gamma_1 = \alpha_0 + \gamma_0 - \frac{[2\alpha_0+\gamma_0(1+\beta_0)]^2+1}{2[2\alpha_0+\gamma_0(1+\beta_0)]} = \alpha_0 + \frac{2\gamma_0[2\alpha_0+\gamma_0(1+\beta_0)]-[2\alpha_0+\gamma_0(1+\beta_0)]^2-1}{2[2\alpha_0+\gamma_0(1+\beta_0)]}$$

$$= \alpha_0 + \frac{4\alpha_0\gamma_0+2\gamma_0^2(1+\beta_0)-4\alpha_0^2-\gamma_0^2(1+\beta_0)^2-4\alpha_0\gamma_0(1+\beta_0)-1}{2[2\alpha_0+\gamma_0(1+\beta_0)]} = \alpha_0 + \frac{-4\alpha_0^2-4\alpha_0\beta_0\gamma_0+\gamma_0^2(1+\beta_0)(1-\beta_0)-1}{2[2\alpha_0+\gamma_0(1+\beta_0)]}$$

$$= \alpha_0 - \frac{2\alpha_0^2+2\alpha_0\beta_0\gamma_0}{2\alpha_0+\gamma_0(1+\beta_0)} = \frac{\alpha_0\gamma_0(1-\beta_0)}{2\alpha_0+\gamma_0(1+\beta_0)} = \frac{\alpha_0(1-\beta_0)/(1+\beta_0)}{1+2\alpha_0\sqrt{(1-\beta_0)/(1+\beta_0)}}. \quad (13)$$

Consequently,

$$\frac{\mathcal{E}_1}{\mathcal{E}_0} = \frac{(1-\beta_0)/(1+\beta_0)}{1+2(\mathcal{E}_0/Mc^2)\sqrt{(1-\beta_0)/(1+\beta_0)}}. \quad (14)$$

If the mirror happens to be massive, $\mathcal{E}_0/Mc^2 \to 0$, in which case the above equation yields the standard Doppler-shift formula for the ratio of the reflected to incident energies (or frequencies). If the initial mirror velocity happens to be zero, then $\mathcal{E}_1 = \mathcal{E}_0/[1 + 2(\mathcal{E}_0/Mc^2)]$ reveals the loss of optical energy upon reflection from a mirror with a finite mass.