Problem 1)

a) In free space, Maxwell's first equation is $\boldsymbol{\nabla}\cdot(\varepsilon_0\boldsymbol{E})=0$. Application to the E-field of the plane-wave yields $\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=0$. Consequently, $\boldsymbol{k}\cdot\boldsymbol{E}_0=0$. This is the general relation between the k-vector and the magnitude $\boldsymbol{E}_0$ of the plane-wave's E-field.

b) In free space, Maxwell's fourth equation is $\boldsymbol{\nabla}\cdot(\mu_0\boldsymbol{H})=0$. Application to the H-field of the plane-wave yields $\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=0$. Consequently, $\boldsymbol{k}\cdot\boldsymbol{H}_0=0$. This is the general relation between the k-vector and the magnitude $\boldsymbol{H}_0$ of the plane-wave's E-field.

c) Maxwell's second equation in free space is $\nabla\times\boldsymbol{H}=\varepsilon_0\,\partial\boldsymbol{E}/\partial t$. Substitution for $\boldsymbol{E}$ and $\boldsymbol{H}$ from the plane-wave expressions yields

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=-\mathrm{i}\omega\varepsilon_0\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\quad\rightarrow\quad\boldsymbol{k}\times\boldsymbol{H}_0=-\varepsilon_0\omega\boldsymbol{E}_0.$$

d) Maxwell's third equation in free space is $\nabla\times\boldsymbol{E}=-\mu_0\,\partial\boldsymbol{H}/\partial t$. Substitution for $\boldsymbol{E}$ and $\boldsymbol{H}$ from the plane-wave expressions yields

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=\mathrm{i}\omega\mu_0\boldsymbol{H}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\quad\rightarrow\quad\boldsymbol{k}\times\boldsymbol{E}_0=\mu_0\omega\boldsymbol{H}_0.$$

e) From part (c) we know that $\boldsymbol{E}_0=-\boldsymbol{k}\times\boldsymbol{H}_0/(\varepsilon_0\omega)$. Substitution in the result obtained in part (d) then yields

$$-\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{H}_0)/(\varepsilon_0\omega)=\mu_0\omega\boldsymbol{H}_0\quad\rightarrow\quad(\boldsymbol{k}\cdot\boldsymbol{H}_0)\boldsymbol{k}-(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{H}_0=-\mu_0\varepsilon_0\omega^2\boldsymbol{H}_0.$$

Now, in part (b) we found that $\boldsymbol{k}\cdot\boldsymbol{H}_0=0$. Therefore, the first term on the left-hand side of the preceding equation disappears, and we are left with $(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{H}_0=\mu_0\varepsilon_0\omega^2\boldsymbol{H}_0$. Dropping $\boldsymbol{H}_0$ from both sides of this equation yields

$$\boldsymbol{k}\cdot\boldsymbol{k}=(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')\cdot(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')=(k'^2-k''^2)+2\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{k}''=\mu_0\varepsilon_0\omega^2=(\omega/c)^2.$$

This is the general relation between the wave-vector $\boldsymbol{k}$ and the frequency ω of a plane-wave in free space.

Problem 2)

a) Symmetry dictates that the E-field be radial and independent of the azimuthal and vertical coordinates φ and z, that is, $\boldsymbol{E}(r,\varphi,z)=E_r(r)\hat{\boldsymbol{r}}$. Now, imagine a cylindrical surface of radius r and height h centered on the z-axis, as shown in figure (a) below. Application to this cylinder of the integral form of Maxwell's first equation $\oint_{\text{surface}}\varepsilon_0\boldsymbol{E}\cdot d\boldsymbol{s}=\int_{\text{volume}}\rho_{\text{free}}dv$ yields

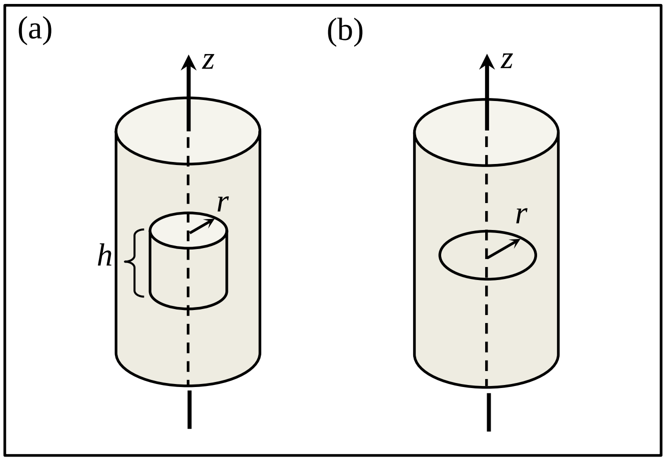
$$2\pi rh\varepsilon_0E_r(r)=\begin{cases}\pi r^2h\rho_0; & r\le R\\ \pi R^2h\rho_0; & r\ge R\end{cases}\quad\rightarrow\quad E_r(r)=\begin{cases}\rho_0r/(2\varepsilon_0); & r\le R,\\ \rho_0R^2/(2\varepsilon_0r); & r\ge R.\end{cases}$$

b) The current-density is the product of charge-density and velocity, that is, $\boldsymbol{J}=\rho_0v_0\hat{\boldsymbol{z}}$.

c) Symmetry dictates that the H-field be azimuthal and independent of the azimuthal and vertical coordinates φ and z, that is, $\boldsymbol{H}(r,\varphi,z)=H_\varphi(r)\hat{\boldsymbol{\varphi}}$. Now, imagine a circular loop of radius r

centered on the z-axis, as shown in figure (b) below. Application to this loop of the integral form of Maxwell's second equation $\oint_{\text{loop}} \boldsymbol{H}\cdot d\boldsymbol{\ell} = \int_{\text{surface}} \boldsymbol{J}\cdot d\boldsymbol{s}$ yields

$$2\pi r H_\varphi(r) = \begin{cases} \pi r^2 \rho_0 v_0; & r \le R \\ \pi R^2 \rho_0 v_0; & r \ge R \end{cases} \quad \rightarrow \quad H_\varphi(r) = \begin{cases} ½\rho_0 v_0 r; & r \le R, \\ ½\rho_0 v_0 R^2/r\,; & r \ge R. \end{cases}$$



d) The Lorentz force-density is given by

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{r},t) &= \rho(\boldsymbol{r},t)[\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{v}(\boldsymbol{r},t)\times\boldsymbol{B}(\boldsymbol{r},t)] = \rho_0\big(E_r\hat{\boldsymbol{r}} + \mu_0 v_0 \hat{\boldsymbol{z}}\times H_\varphi \hat{\boldsymbol{\varphi}}\big) \\ &= \rho_0\big(E_r\hat{\boldsymbol{r}} - \mu_0 v_0 H_\varphi \hat{\boldsymbol{r}}\big) = \rho_0[\rho_0 r/(2\varepsilon_0) - (\mu_0 v_0)(½\rho_0 v_0 r)]\hat{\boldsymbol{r}} \\ &= ½\mu_0(c^2 - v_0^2)\rho_0^2\, r\hat{\boldsymbol{r}}. \end{aligned}$$

It is readily observed that the radially outward push of the electric force is proportional to $c^2 = (\mu_0\varepsilon_0)^{-1}$, whereas the radially inward pull of the magnetic force is proportional to v_0^2. These two forces will cancel out only if $v_0 = c$.

Problem 3)

a) The polarization $\boldsymbol{P}$ has units of *coulomb*/m^2. Since the units of current are *ampere*, and since, by definition, electrical current I is the rate-of-flow of charge ΔQ during the time interval Δt, that is $I = \Delta Q/\Delta t$ in the limit when $\Delta t \to 0$, the units of charge are *coulomb*=*ampere*·*sec*. Therefore, the units of $\boldsymbol{P}$ are *ampere*·*sec*/m^2.

b) The magnetization $\boldsymbol{M}$ has units of *weber*/m^2. Since, by definition, $\boldsymbol{B} = \mu_0\boldsymbol{H} + \boldsymbol{M}$, the units of $\boldsymbol{M}$ must be the same as those of $\boldsymbol{B}$. The Lorentz force law states that $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B})$, where $\boldsymbol{F}$ is force (units: *newton*), q is electrical charge (units: *coulomb*), $\boldsymbol{E}$ is electric field (units: *volt*/*m*), $\boldsymbol{V}$ is velocity (units: *m*/*sec*), and $\boldsymbol{B}$ is magnetic induction (units: *weber*/m^2). We may thus write: *newton*=*coulomb*·(*m*/*sec*)·(*weber*/m^2). Now, from newton's law $\boldsymbol{F} = m\boldsymbol{a}$, we have *newton*=*kg*·*m*/sec^2. Also, from $I = \Delta Q/\Delta t$, we have *coulomb*=*ampere*·*sec*. Consequently,

$$weber/m^2 = (kg\cdot m/sec^2)/(ampere\cdot m) = kg/(ampere\cdot sec^2).$$

c) The electrical resistance R has units of *ohm*. According to Ohm's law, the voltage V and the current I of a resistor R are related via $V = RI$. Therefore, R has units of *volt*/*ampere*. From the

Lorentz force law cited in part (b), we know that $newton = coulomb \cdot (volt/m)$, that is, $volt = newton \cdot m/coulomb = kg \cdot m^2/(ampere \cdot sec^3)$. Consequently, $ohm = kg \cdot m^2/(ampere^2 \cdot sec^3)$.

d) The units of capacitance C are *farad*. The relation between electrical charge on the plates of a capacitor and the voltage difference between those plates is $Q = CV$. Consequently,

$$farad = coulomb/volt = ampere \cdot sec/(newton \cdot m/coulomb) = ampere^2 \cdot \sec^4/(kg \cdot m^2).$$

e) The units of inductance L are *henry*. The relation between the voltage V and the current I of a solenoid are $V = L(dI/dt)$. Therefore, the units of L must be $volt \cdot sec/ampere$. We know from the Lorentz force law that $volt = newton \cdot m/coulomb$. Consequently,

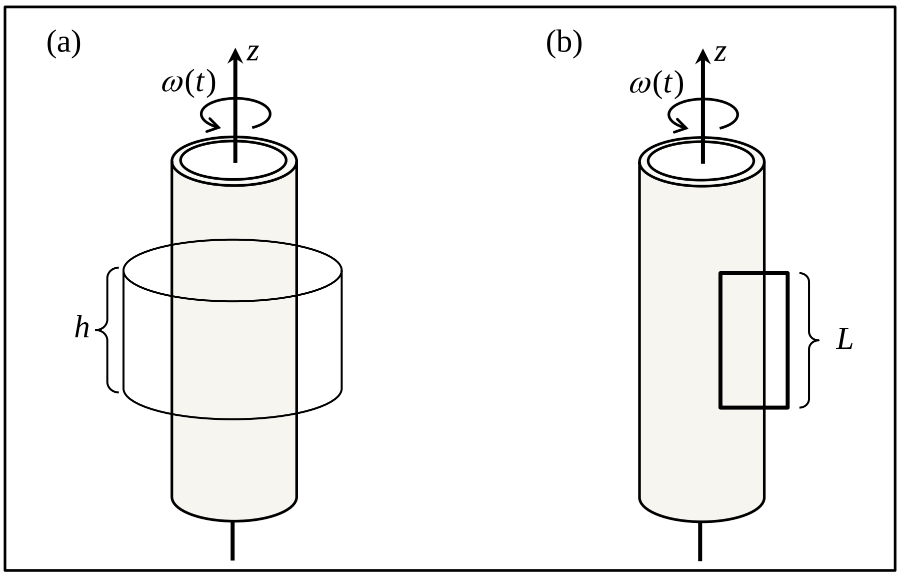
$$henry = newton \cdot m \cdot sec/(coulomb \cdot ampere) = kg \cdot m^2/(ampere^2 \cdot sec^2).$$

f) The electromagnetic momentum-density $\boldsymbol{p}_{EM} = \boldsymbol{E} \times \boldsymbol{H}/c^2$ has units of the E-field times the units of the H-field divided by the units of velocity squared, i.e., $(volt/m) \cdot (ampere/m)/(m^2/sec^2)$. From the Lorentz force law we know that $volt/m = newton/coulomb = kg \cdot m/(sec^2 \cdot coulomb)$. Therefore, the units of $\boldsymbol{p}_{EM}$ are $kg/(m^2 \cdot sec)$.

Problem 4)

a) The surface charge-density σ_s remains constant at all times as the cylinder spins up; in other words, the rotation of the cylinder does *not* affect the surface charge-density. Invoking the symmetry of the problem, one can argue that the E-field has only a radial component which does not vary with φ and z, that is, $\boldsymbol{E}(\boldsymbol{r}, t) = E_r(r)\hat{\boldsymbol{r}}$. Application of Gauss's law (i.e., Maxwell's first equation) to a coaxial cylinder of radius r and height h, as shown in figure (a) below, then yields

$$2\pi r h \varepsilon_0 E_r(r) = \begin{cases} 0; & r < R \\ 2\pi R h \sigma_s; & r > R \end{cases} \quad \rightarrow \quad \boldsymbol{E}(\boldsymbol{r}, t) = \begin{cases} 0; & r < R, \\ \sigma_s R/(r\varepsilon_0)\,; & r > R. \end{cases}$$



b) The surface current-density is the product of the surface charge-density and the linear velocity of the cylinder, that is, $\boldsymbol{J}_s(t) = \sigma_s R\omega(t)\hat{\boldsymbol{\varphi}}$. Assuming the cylinder spins up slowly, the magnetic field outside the cylinder can be shown to be negligible. Inside the cylinder, the field is uniform and aligned with the z-axis. Application of the integral form of Ampere's law (i.e., Maxwell's second equation $\oint_{\text{loop}} \boldsymbol{H} \cdot d\boldsymbol{\ell} = \int_{\text{surface}} \boldsymbol{J} \cdot d\boldsymbol{s}$) to the rectangular loop shown in figure (b) reveals that $\boldsymbol{H}(\boldsymbol{r},t) = \sigma_s R\omega(t)\hat{\boldsymbol{z}}$, when $r < R$.

c) The energy-density of the magnetic field is $\mathcal{E}(\boldsymbol{r},t) = ½\mu_0 H^2(\boldsymbol{r},t) = ½\mu_0\sigma_s^2 R^2\omega^2(t)$. Since the volume associated with a unit length of the cylinder is πR^2, in the steady state when the cylinder reaches its constant angular velocity ω_0, the stored energy per unit length will be $½\pi\mu_0\sigma_s^2 R^4\omega_0^2$.

d) Consider a perpendicular cross-section of the cylinder, namely, a circle of radius R centered on the cylinder axis. We apply Faraday's law (i.e., Maxwell's third equation) to this loop in order to determine the induced E-field that opposes the angular acceleration of the cylinder. The integral form of Faraday's law is written

$$\oint_{\text{loop}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_{\text{surface}} \boldsymbol{B}(\boldsymbol{r},t) \cdot d\boldsymbol{s}$$

Application of the above equation to the circular loop of radius R yields

$$2\pi R E_\varphi(r = R,t) = -(\pi R^2)[\mu_0\sigma_s R\omega'(t)] \quad \rightarrow \quad E_\varphi(r = R,t) = -½\mu_0\sigma_s R^2\omega'(t).$$

e) The work done per unit length of the cylinder during the spin-up process is evaluated by integrating (from $t = 0$ to ∞) the product of the electromagnetic force and velocity, namely,

$$(2\pi R)\boldsymbol{f}(r = R,t) \cdot \boldsymbol{v}(t) = (2\pi R)\sigma_s E_\varphi(r = R,t)\hat{\boldsymbol{\varphi}} \cdot R\omega(t)\hat{\boldsymbol{\varphi}}.$$

The mechanical energy needed to spin the cylinder from $\omega(0) = 0$ to $\omega(\infty) = \omega_0$ is thus found to be

$$-\int_0^\infty (2\pi R)\boldsymbol{f}(r = R,t) \cdot \boldsymbol{v}(t)dt = \pi\mu_0\sigma_s^2 R^4 \int_0^\infty \omega'(t)\omega(t)dt = ½\pi\mu_0\sigma_s^2 R^4\omega_0^2.$$

f) The total mechanical energy needed to bring up the cylinder from its initial (stationary) state to its final state — where it spins at the constant angular velocity ω_0 — is seen from parts (c) and (e) above to be exactly equal to the energy stored in the magnetic field within the cylinder.