Problem 1)

a) $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = (\boldsymbol{\alpha}' + \mathrm{i}\boldsymbol{\alpha}'')\cdot(\boldsymbol{\beta}' + \mathrm{i}\boldsymbol{\beta}'') = \overbrace{(\boldsymbol{\alpha}'\cdot\boldsymbol{\beta}' - \boldsymbol{\alpha}''\cdot\boldsymbol{\beta}'')}^{\text{Real}} + \mathrm{i}\overbrace{(\boldsymbol{\alpha}'\cdot\boldsymbol{\beta}'' + \boldsymbol{\alpha}''\cdot\boldsymbol{\beta}')}^{\text{Imaginary}}.$

b) $\boldsymbol{\alpha}\times\boldsymbol{\beta} = (\boldsymbol{\alpha}' + \mathrm{i}\boldsymbol{\alpha}'')\times(\boldsymbol{\beta}' + \mathrm{i}\boldsymbol{\beta}'') = (\boldsymbol{\alpha}'\times\boldsymbol{\beta}' - \boldsymbol{\alpha}''\times\boldsymbol{\beta}'') + \mathrm{i}(\boldsymbol{\alpha}'\times\boldsymbol{\beta}'' + \boldsymbol{\alpha}''\times\boldsymbol{\beta}').$

c) $\boldsymbol{\gamma}\times\boldsymbol{\gamma} = (\boldsymbol{\gamma}' + \mathrm{i}\boldsymbol{\gamma}'')\times(\boldsymbol{\gamma}' + \mathrm{i}\boldsymbol{\gamma}'') = (\underbrace{\boldsymbol{\gamma}'\times\boldsymbol{\gamma}'}_{0} - \underbrace{\boldsymbol{\gamma}''\times\boldsymbol{\gamma}''}_{0}) + \mathrm{i}(\boldsymbol{\gamma}'\times\boldsymbol{\gamma}'' + \underbrace{\boldsymbol{\gamma}''\times\boldsymbol{\gamma}'}_{-\boldsymbol{\gamma}'\times\boldsymbol{\gamma}''}) = 0 + \mathrm{i}0.$

d) $$\begin{aligned}\boldsymbol{\alpha}\cdot(\boldsymbol{\beta}\times\boldsymbol{\gamma}) &= (\boldsymbol{\alpha}' + \mathrm{i}\boldsymbol{\alpha}'')\cdot[(\boldsymbol{\beta}' + \mathrm{i}\boldsymbol{\beta}'')\times(\boldsymbol{\gamma}' + \mathrm{i}\boldsymbol{\gamma}'')]\\
&= (\boldsymbol{\alpha}' + \mathrm{i}\boldsymbol{\alpha}'')\cdot[(\boldsymbol{\beta}'\times\boldsymbol{\gamma}' - \boldsymbol{\beta}''\times\boldsymbol{\gamma}'') + \mathrm{i}(\boldsymbol{\beta}'\times\boldsymbol{\gamma}'' + \boldsymbol{\beta}''\times\boldsymbol{\gamma}')]\\
&= [\boldsymbol{\alpha}'\cdot(\boldsymbol{\beta}'\times\boldsymbol{\gamma}' - \boldsymbol{\beta}''\times\boldsymbol{\gamma}'') - \boldsymbol{\alpha}''\cdot(\boldsymbol{\beta}'\times\boldsymbol{\gamma}'' + \boldsymbol{\beta}''\times\boldsymbol{\gamma}')] \quad \leftarrow \text{Real}\\
&\quad + \mathrm{i}[\boldsymbol{\alpha}'\cdot(\boldsymbol{\beta}'\times\boldsymbol{\gamma}'' + \boldsymbol{\beta}''\times\boldsymbol{\gamma}') + \boldsymbol{\alpha}''\cdot(\boldsymbol{\beta}'\times\boldsymbol{\gamma}' - \boldsymbol{\beta}''\times\boldsymbol{\gamma}'')]. \quad \leftarrow \text{Imaginary}\end{aligned}$$

Problem 2)

a) $$\begin{aligned}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) &= \boldsymbol{\nabla}\times\{\mathrm{i}\boldsymbol{k}_1\psi_0\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega_1 t)]\}\\
&= \mathrm{i}^2\underbrace{(\boldsymbol{k}_1\times\boldsymbol{k}_1)}_{0}\psi_0\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega_1 t)] = 0.\end{aligned}$$

b) $$\begin{aligned}\boldsymbol{\nabla}\cdot(\psi\boldsymbol{A}) &= \boldsymbol{\nabla}\cdot(\psi_0\boldsymbol{A}_0\exp\{\mathrm{i}[(\boldsymbol{k}_1 + \boldsymbol{k}_2)\cdot\boldsymbol{r} - (\omega_1 + \omega_2)t]\})\\
&= \mathrm{i}\psi_0(\boldsymbol{k}_1 + \boldsymbol{k}_2)\cdot\boldsymbol{A}_0\exp\{\mathrm{i}[(\boldsymbol{k}_1 + \boldsymbol{k}_2)\cdot\boldsymbol{r} - (\omega_1 + \omega_2)t]\}\\
&= \{\mathrm{i}\boldsymbol{k}_1\psi_0\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega_1 t)]\}\cdot\boldsymbol{A}_0\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega_2 t)]\\
&\quad + \psi_0\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega_1 t)]\,\{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{A}_0\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega_2 t)]\}\\
&= (\boldsymbol{\nabla}\psi)\cdot\boldsymbol{A} + \psi(\boldsymbol{\nabla}\cdot\boldsymbol{A}).\end{aligned}$$

c) $$\begin{aligned}\boldsymbol{\nabla}\times(\psi\boldsymbol{B}) &= \boldsymbol{\nabla}\times(\psi_0\boldsymbol{B}_0\exp\{\mathrm{i}[(\boldsymbol{k}_1 + \boldsymbol{k}_3)\cdot\boldsymbol{r} - (\omega_1 + \omega_3)t]\})\\
&= \mathrm{i}\psi_0(\boldsymbol{k}_1 + \boldsymbol{k}_3)\times\boldsymbol{B}_0\exp\{\mathrm{i}[(\boldsymbol{k}_1 + \boldsymbol{k}_3)\cdot\boldsymbol{r} - (\omega_1 + \omega_3)t]\}\\
&= \{\mathrm{i}\boldsymbol{k}_1\psi_0\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega_1 t)]\}\times\boldsymbol{B}_0\exp[\mathrm{i}(\boldsymbol{k}_3\cdot\boldsymbol{r} - \omega_3 t)]\\
&\quad + \psi_0\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega_1 t)]\,\{\mathrm{i}\boldsymbol{k}_3\times\boldsymbol{B}_0\exp[\mathrm{i}(\boldsymbol{k}_3\cdot\boldsymbol{r} - \omega_3 t)]\}\\
&= (\boldsymbol{\nabla}\psi)\times\boldsymbol{B} + \psi\boldsymbol{\nabla}\times\boldsymbol{B}.\end{aligned}$$

d) $$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) &= \boldsymbol{\nabla}\cdot(\boldsymbol{A}_0\times\boldsymbol{B}_0\exp\{\mathrm{i}[(\boldsymbol{k}_2 + \boldsymbol{k}_3)\cdot\boldsymbol{r} - (\omega_2 + \omega_3)t]\})\\
&= \mathrm{i}(\boldsymbol{k}_2 + \boldsymbol{k}_3)\cdot(\boldsymbol{A}_0\times\boldsymbol{B}_0)\exp\{\mathrm{i}[(\boldsymbol{k}_2 + \boldsymbol{k}_3)\cdot\boldsymbol{r} - (\omega_2 + \omega_3)t]\}\\
&= \mathrm{i}\boldsymbol{k}_2\cdot(\boldsymbol{A}_0\times\boldsymbol{B}_0)\exp\{\mathrm{i}[(\boldsymbol{k}_2 + \boldsymbol{k}_3)\cdot\boldsymbol{r} - (\omega_2 + \omega_3)t]\}\\
&\quad + \mathrm{i}\boldsymbol{k}_3\cdot(\boldsymbol{A}_0\times\boldsymbol{B}_0)\exp\{\mathrm{i}[(\boldsymbol{k}_2 + \boldsymbol{k}_3)\cdot\boldsymbol{r} - (\omega_2 + \omega_3)t]\}\\
&= (\mathrm{i}\boldsymbol{k}_2\times\boldsymbol{A}_0)\cdot\boldsymbol{B}_0\exp\{\mathrm{i}[(\boldsymbol{k}_2 + \boldsymbol{k}_3)\cdot\boldsymbol{r} - (\omega_2 + \omega_3)t]\}\\
&\quad - (\mathrm{i}\boldsymbol{k}_3\times\boldsymbol{B}_0)\cdot\boldsymbol{A}_0\exp\{\mathrm{i}[(\boldsymbol{k}_2 + \boldsymbol{k}_3)\cdot\boldsymbol{r} - (\omega_2 + \omega_3)t]\}\\
&= \{\mathrm{i}\boldsymbol{k}_2\times\boldsymbol{A}_0\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega_2 t)]\}\cdot\boldsymbol{B}_0\exp[\mathrm{i}(\boldsymbol{k}_3\cdot\boldsymbol{r} - \omega_3 t)]\\
&\quad - \{\mathrm{i}\boldsymbol{k}_3\times\boldsymbol{B}_0\exp[\mathrm{i}(\boldsymbol{k}_3\cdot\boldsymbol{r} - \omega_3 t)]\}\cdot\boldsymbol{A}_0\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega_2 t)]\\
&= (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\boldsymbol{B} - (\boldsymbol{\nabla}\times\boldsymbol{B})\cdot\boldsymbol{A} = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}).\end{aligned}$$

Problem 3)

a) The magnetic flux through the circular loop is the integral of the B-field over the surface are of the loop. The field is uniform and confined to an area A defined by the pole-pieces; therefore, $\Phi(t) = AB_0 \cos(\omega t)$.

b) Using Stokes' theorem, Faraday's law, $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, may be written in integral form as follows:

$$\oint_{loop} \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_{surface} \boldsymbol{B} \cdot d\boldsymbol{s} = -\frac{d\Phi(t)}{dt} = AB_0\omega \sin(\omega t).$$

Symmetry dictates that the E-field be uniform around the circle and directed along the azimuthal axis $\hat{\boldsymbol{\varphi}}$. Therefore,

$$2\pi\rho E_\varphi = AB_0\omega \sin(\omega t) \quad \rightarrow \quad \boldsymbol{E} = \frac{AB_0\omega \sin(\omega t)}{2\pi\rho}\,\hat{\boldsymbol{\varphi}}.$$

c) The induced voltage in the loop is $V(t) = \oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = AB_0\omega \sin(\omega t)$. Considering that, in accordance with Ohm's law, $V(t) = R\,I(t)$, we will have $I(t) = (AB_0\omega/R)\sin(\omega t)$.

Problem 4)

a) Electric field $\boldsymbol{E}$ $[volt/meter]$: Use Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$ and the Lorentz force law $\boldsymbol{F} = q\boldsymbol{E}$. The units of force $\boldsymbol{F}$ are thus $[kg \cdot meter/\sec^2]$, and the units of the electric field are $[kg \cdot meter/(coulomb \cdot \sec^2)]$ or $[kg \cdot meter/(ampere \cdot \sec^3)]$.

b) Magnetic induction $\boldsymbol{B}$ $[weber/m^2]$: Use Faraday's law, $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, in conjunction with the units of $\boldsymbol{E}$ determined in part (a). The curl operator does differentiation with respect to spatial coordinates; therefore, the units of $\boldsymbol{\nabla} \times \boldsymbol{E}$ are those of $\boldsymbol{E}$ divided by the length unit, $[meter]$. On the right-hand-side of the equation, the time-derivative of $\boldsymbol{B}$ has the units of $\boldsymbol{B}$ divided by those of time, $[second]$. Consequently, $[weber/m^2] = [kg/(ampere \cdot \sec^2)]$.

c) Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ $[volt \cdot ampere/m^2]$: The units of $\boldsymbol{H}$ are $[ampere/meter]$, and the units of $\boldsymbol{E}$ were found in part (a) to be $[kg \cdot meter/(ampere \cdot \sec^3)]$. Multiplying the two, we find the units of $\boldsymbol{S}$ to be $[volt \cdot ampere/m^2] = [kg/sec^3]$. Note that $\boldsymbol{S}$ is expected to have the units of energy per unit area per unit time. Energy, however, has the units of force times displacement, that is, $[kg \cdot meter^2/sec^2]$. Dividing by $[meter^2 \cdot sec]$ then yields the units of S as before, namely, $[kg/sec^3]$.

d) Permittivity of free space ε_0 $[farad/meter]$: The charge Q and the voltage V of a capacitor are related via $Q = CV$. The capacitance C has units of $[farad]$, the units of Q are $[coulomb] = [ampere \cdot sec]$, and the units of V are $[kg \cdot meter^2/(ampere \cdot \sec^3)]$; see part (a). Consequently, the units of ε_0 are $[farad/meter] = [ampere^2 \cdot \sec^4/(kg \cdot meter^3)]$.

e) Permeability of free space μ_0 $[henry/meter]$: Use the fact that in free space $\boldsymbol{B} = \mu_0\boldsymbol{H}$. The units of $\boldsymbol{H}$ are $[ampere/meter]$; those of $\boldsymbol{B}$ were found in part (b) to be $[kg/(ampere \cdot \sec^2)]$. Consequently, the units of μ_0 are $[henry/meter] = [kg \cdot meter/(ampere^2 \cdot \sec^2)]$.
