Problem 1) a) The linear velocity of the spherical surface is $\boldsymbol{V}(\rho = R, \theta, \phi) = (R\sin\theta)\omega\,\hat{\boldsymbol{\phi}}$. Therefore, the surface current density is $\boldsymbol{J}_s(R,\theta,\phi,t) = \sigma_s \boldsymbol{V}(R,\theta,\phi) = (R\omega\sigma_s \sin\theta)\hat{\boldsymbol{\phi}}$. The units of $\boldsymbol{J}_s$ are the units of R [m] times the units of ω [sec^{-1}] times the units of σ_s [coulomb/m^2], namely, [ampere/m].

b) In spherical coordinates, the divergence of the vector field $\boldsymbol{J}_s$ whose only component is along the ϕ-axis, is given by $\boldsymbol{\nabla}\cdot\boldsymbol{J}_s = \frac{1}{R\sin\theta}\frac{\partial J_{s\phi}}{\partial\phi} = 0$. Since the surface-charge-density σ_s has no time-dependence, its derivative with respect to time is zero, that is, $\frac{\partial\sigma_s}{\partial t} = 0$. Clearly, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_s + \frac{\partial\sigma_s}{\partial t} = 0$.

Problem 2)

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= \text{Real}\{\boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\}\\ &= \exp(-\boldsymbol{k}''\cdot\boldsymbol{r})\,\text{Real}\{(\boldsymbol{E}_0' + i\boldsymbol{E}_0'')\exp[i(\boldsymbol{k}'\cdot\boldsymbol{r} - \omega t)]\}\\ &= \exp(-\boldsymbol{k}''\cdot\boldsymbol{r})\,\text{Real}\{(\boldsymbol{E}_0' + i\boldsymbol{E}_0'')[\cos(\boldsymbol{k}'\cdot\boldsymbol{r} - \omega t) + i\sin(\boldsymbol{k}'\cdot\boldsymbol{r} - \omega t)]\}\\ &= \exp(-\boldsymbol{k}''\cdot\boldsymbol{r})\,[\boldsymbol{E}_0'\cos(\boldsymbol{k}'\cdot\boldsymbol{r} - \omega t) - \boldsymbol{E}_0''\sin(\boldsymbol{k}'\cdot\boldsymbol{r} - \omega t)].\end{aligned}$$

a) As a function of time, the field oscillates at the angular frequency ω.

b) The factor $\exp(-\boldsymbol{k}''\cdot\boldsymbol{r})$ is responsible for the decay of the field amplitude. The *E*-field thus decays along the direction of $\boldsymbol{k}''$ at a rate determined by the magnitude k'' of the vector $\boldsymbol{k}''$. The planes of constant amplitude are perpendicular to $\boldsymbol{k}''$.

c) The phase of the *E*-field is the argument of the sine and cosine functions, namely, $\boldsymbol{k}'\cdot\boldsymbol{r} - \omega t$. At any given time *t*, the phase is the same for all the points $\boldsymbol{r}$ in a plane perpendicular to $\boldsymbol{k}'$. Thus, within each and every plane that is perpendicular to $\boldsymbol{k}'$, the *E*-field has the same phase at any given instant *t* of time. If two such planes are separated by a distance of $2\pi/k'$ (along the direction of $\boldsymbol{k}'$), the phase difference between the two planes will be

$$(\boldsymbol{k}'\cdot\boldsymbol{r}_1 - \omega t) - (\boldsymbol{k}'\cdot\boldsymbol{r}_2 - \omega t) = \boldsymbol{k}'\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2) = k'(2\pi/k') = 2\pi.$$

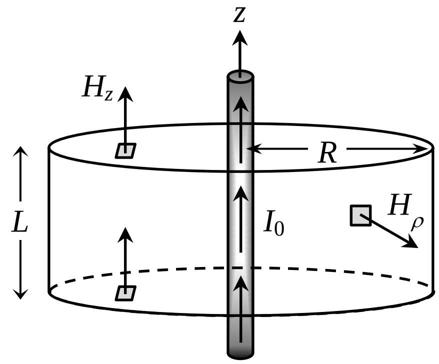
Therefore, at any given time *t*, the *E*-field amplitude is the same in all the planes that are perpendicular to $\boldsymbol{k}'$ and are separated from each other (along the direction of $\boldsymbol{k}'$) by a distance of $2\pi/k'$.

Consider an arbitrary point in the three-dimensional Euclidean space whose position vector $\boldsymbol{r}$ is aligned with the vector $\boldsymbol{k}'$. If the length of this vector is increased by Δr while the time is advanced by Δt, the phase of the *E*-field will change by $k'\Delta r - \omega\Delta t$. The change of phase will be zero if $\Delta r/\Delta t = \omega/k'$. The phase velocity of the plane-wave is, therefore, $V_{\text{phase}} = \omega/k'$.

d) The polarization state of the plane-wave is determined by $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$. The beam is linearly polarized if $\boldsymbol{E}_0' = 0$, or $\boldsymbol{E}_0'' = 0$, or $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ are parallel to each other. The beam is circularly polarized if $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ have equal lengths *and* are perpendicular to each other.

Problem 3) Consider a cylindrical can of arbitrary radius R and arbitrary length L centered on the wire, as shown in the figure below. We apply the integral form of Maxwell's 4th equation, $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$, to this can. Using Gauss's theorem, the integral form is found to be: $\oint \boldsymbol{B}\cdot d\boldsymbol{s}=0$. In the absence of magnetism, we have $\boldsymbol{M}=0$ and $\boldsymbol{B}=\mu_o\boldsymbol{H}$. Consequently, Maxwell's 4th equation demands that $\oint \boldsymbol{H}\cdot d\boldsymbol{s}=0$.

On the closed surface of the cylindrical can, only two components of the H-field contribute to the surface integral: (i) on the top and bottom facets, H_z has nonzero integrals; (ii) on the cylindrical surface, H_ρ makes a nonzero contribution to the integral. However, symmetry indicates that the contribution of H_z to the top facet is exactly cancelled out by its contribution to the bottom facet– because the value of H_z (whatever it may be) cannot depend on the z-coordinate. As for H_ρ, its magnitude must be the same everywhere on the cylindrical surface, again because of symmetry; its contribution to the surface integral will thus be $2\pi RLH_\rho$. The total integral of the H-field over the surface of the can is, therefore, $2\pi RLH_\rho$, which must be zero in accordance with Maxwell's 4th equation. We conclude that $H_\rho(\rho,\phi,z)$ must be zero everywhere.



Problem 4)

a) $\rho^{(e)}_{\text{bound}}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t)=-\frac{\partial P_z}{\partial z}=-P_0\kappa\,\delta(y)\cos(\kappa z-\omega t)$.

$\boldsymbol{J}^{(e)}_{\text{bound}}(\boldsymbol{r},t)=\frac{\partial \boldsymbol{P}}{\partial t}=-P_0\omega\delta(y)\cos(\kappa z-\omega t)\hat{\boldsymbol{z}}$.

$\boldsymbol{\nabla}\cdot\boldsymbol{J}+\frac{\partial\rho}{\partial t}=\frac{\partial J_z}{\partial z}+\frac{\partial\rho}{\partial t}=P_0\omega\kappa\delta(y)\sin(\kappa z-\omega t)-P_0\kappa\omega\,\delta(y)\sin(\kappa z-\omega t)=0$.

b) $\rho^{(e)}_{\text{bound}}(\boldsymbol{r},t)=0$.

$\boldsymbol{J}^{(e)}_{\text{bound}}(\boldsymbol{r},t)=\mu_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}=\mu_o^{-1}\left(\frac{\partial M_y}{\partial x}\hat{\boldsymbol{z}}-\frac{\partial M_y}{\partial z}\hat{\boldsymbol{x}}\right)=\mu_o^{-1}M_0\kappa\,\delta(y)\sin(\kappa z-\omega t)\hat{\boldsymbol{x}}$.

$\boldsymbol{\nabla}\cdot\boldsymbol{J}+\frac{\partial\rho}{\partial t}=\frac{\partial J_x}{\partial x}+\frac{\partial\rho}{\partial t}=0$.

Problem 5) Imagine a cylindrical can of arbitrary length L and radius R, where $R_1<R<R_2$, centered on the z-axis. The integral of the E-field over the closed cylindrical surface must be zero because (i) inside the metallic shell there cannot be any E-field, and (ii) E_z must vanish at the top and bottom facets of the can (because of symmetry). Therefore, the integral form of Maxwell's 1st equation, namely, $\oint \boldsymbol{D}\cdot d\boldsymbol{s}=Q_{\text{total}}$, requires that the total free charge Q_{total} contained within the can must be zero.

Since the wire has a charge of $\lambda_0 L$ inside the can, the inner surface of the metallic shell must have an equal and opposite charge. The surface charge density on the inner surface of the shell is thus $\sigma_1=-(\lambda_0L)/(2\pi R_1L)=-\lambda_0/(2\pi R_1)$ [coulomb/m^2]. Since the shell is initially charge-neutral, the same amount of charge, albeit with opposite sign, must appear on its external surface. Therefore, $\sigma_2=\lambda_0/(2\pi R_2)$ [coulomb/m^2].
