Problem 1) a) The amount of charge flowing per unit time through each and every cross-section perpendicular to the z-axis is given by

$$I_o = \lambda_o V. \quad (1)$$

Note that the units on both sides of the above equation are [coulomb/meter].

b) Application of Maxwell's 1st equation to a cylinder of radius r and unit height along z yields the E-field as $\boldsymbol{E}(r,\phi,z)=\lambda_o\hat{\boldsymbol{r}}/(2\pi\varepsilon_o r)$. Similarly, application of Maxwell's 2nd equation to a circular loop of radius r parallel to the xy-plane yields the H-field as $\boldsymbol{H}(r,\phi,z)=\lambda_o V\hat{\boldsymbol{\phi}}/(2\pi r)$.

c) $$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\varepsilon_o\boldsymbol{\nabla}\cdot\boldsymbol{E}=\varepsilon_o\frac{\partial(rE_r)}{r\partial r}=\varepsilon_o\frac{\partial[r\lambda_o/(2\pi\varepsilon_o r)]}{r\partial r}=\frac{\partial(\lambda_o)}{2\pi r\partial r}=0. \quad (2)$$

$$\boldsymbol{\nabla}\times\boldsymbol{H}=-\frac{\partial H_\phi}{\partial z}\hat{\boldsymbol{r}}+\frac{\partial(rH_\phi)}{r\partial r}\hat{\boldsymbol{z}}=\frac{\partial[r\lambda_o V/(2\pi r)]}{r\partial r}\hat{\boldsymbol{z}}=\frac{\partial(\lambda_o V)}{2\pi r\partial r}\hat{\boldsymbol{z}}=0. \quad (3)$$

Considering that $\boldsymbol{J}_{\text{free}}=0$ in the surrounding space, and that $\boldsymbol{D}=\varepsilon_o\boldsymbol{E}$ is time-independent, the right-hand side of the above equation is equal to $\boldsymbol{J}_{\text{free}}+\partial\boldsymbol{D}/\partial t$. Maxwell's 2nd equation is thus satisfied.

$$\boldsymbol{\nabla}\times\boldsymbol{E}=\frac{\partial E_r}{\partial z}\hat{\boldsymbol{\phi}}-\frac{\partial E_r}{r\partial\phi}\hat{\boldsymbol{z}}=\frac{\partial[\lambda_o/(2\pi\varepsilon_o r)]}{\partial z}\hat{\boldsymbol{\phi}}-\frac{\partial[\lambda_o/(2\pi\varepsilon_o r)]}{r\partial\phi}\hat{\boldsymbol{z}}=0. \quad (4)$$

Since $\boldsymbol{B}=\mu_o\boldsymbol{H}$ is time-independent, the right-hand side of the above equation is equal to $-\partial\boldsymbol{B}/\partial t$ and, therefore, Maxwell's 3rd equation is satisfied.

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=\mu_o\boldsymbol{\nabla}\cdot\boldsymbol{H}=\mu_o\frac{\partial H_\phi}{r\partial\phi}=\mu_o\frac{\partial[\lambda_o V/(2\pi r)]}{r\partial\phi}=0. \quad (5)$$

d) $$\boldsymbol{S}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)=\frac{\lambda_o\hat{\boldsymbol{r}}}{2\pi\varepsilon_o r}\times\frac{\lambda_o V\hat{\boldsymbol{\phi}}}{2\pi r}=\frac{\lambda_o^2 V\hat{\boldsymbol{z}}}{4\pi^2\varepsilon_o r^2}. \quad (6)$$

Electromagnetic energy thus flows along the z-axis within the free space region surrounding the rod. The rate of flow of this energy drops with the square of radial distance from the rod.

Problem 2)

a) $$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\text{free}}\;\rightarrow\;\varepsilon_o\boldsymbol{\nabla}\cdot\boldsymbol{E}=0\;\rightarrow\;\partial E_z/\partial z=\partial[E_o\cos(k_o y-\omega_o t)]/\partial z=0. \quad (1)$$

$$\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\text{free}}+\partial\boldsymbol{D}/\partial t\;\rightarrow\;-\partial H_x/\partial y=\varepsilon_o\partial E_z/\partial t$$

$$\rightarrow\;H_o k_o\sin(k_o y-\omega_o t)=\varepsilon_o E_o\omega_o\sin(k_o y-\omega_o t)\;\rightarrow\;H_o k_o=\varepsilon_o E_o\omega_o. \quad (2)$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t\;\rightarrow\;\partial E_z/\partial y=-\mu_o\partial H_x/\partial t$$

$$\rightarrow\;-E_o k_o\sin(k_o y-\omega_o t)=-\mu_o H_o\omega_o\sin(k_o y-\omega_o t)\;\rightarrow\;E_o k_o=\mu_o H_o\omega_o. \quad (3)$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \;\;\rightarrow\;\; \mu_o\boldsymbol{\nabla}\cdot\boldsymbol{H}=0 \;\;\rightarrow\;\; \partial H_x/\partial x=\partial[H_o\cos(k_o y-\omega_o t)]/\partial x=0. \tag{4}$$

It is seen that Maxwell's 1st and 4th equations are already satisfied. As for the 2nd and 3rd equations, we note that Eq.(2) above yields $E_o/H_o=k_o/(\varepsilon_o\omega_o)$, whereas Eq.(3) yields $E_o/H_o=\mu_o\omega_o/k_o$. Consequently, we must have $k_o/(\varepsilon_o\omega_o)=\mu_o\omega_o/k_o$, which yields $k_o=\omega_o/c$. Substitution into either Eq.(2) or Eq.(3) now reveals that $E_o/H_o=Z_o$.

b) The discontinuity of $D_\perp=\varepsilon_o E_z$ at each surface is equal to the surface charge-density at that surface, that is,

$$\sigma_s(x,y,z=\pm\tfrac{1}{2}d,t)=\mp\,\varepsilon_o E_o\cos(k_o y-\omega_o t). \tag{5}$$

Similarly, the discontinuity of $H_{||}=H_x$ at each surface is equal to the surface current-density at the corresponding surface, with the current's direction being perpendicular to that of the H-field. We thus have

$$\boldsymbol{J}_s(x,y,z=\pm\tfrac{1}{2}d,t)=\mp\,H_o\cos(k_o y-\omega_o t)\hat{\boldsymbol{y}}. \tag{6}$$

c) At each surface, the charge-current continuity equation $\boldsymbol{\nabla}\cdot\boldsymbol{J}+\partial\rho/\partial t=0$ reduces to $\partial J_{sy}/\partial y+\partial\sigma_s/\partial t=0$. With the help of Eqs.(5) and (6), we write the continuity equation as follows:

$$\begin{aligned}\partial J_{sy}/\partial y+\partial\sigma_s/\partial t&=\pm H_o k_o\sin(k_o y-\omega_o t)\mp\varepsilon_o E_o\omega_o\sin(k_o y-\omega_o t)\\&=\pm(H_o k_o-\varepsilon_o E_o\omega_o)\sin(k_o y-\omega_o t)=0.\end{aligned} \tag{7}$$

In the last line of the above equation, we have used Eq.(2) to set $H_o k_o$ equal to $\varepsilon_o E_o\omega_o$.

Problem 3) a) The orbital angular momentum of the revolving particle is $\boldsymbol{\mathcal{L}}=\mathbf{r}\times\mathbf{p}=mVr_o\hat{\mathbf{z}}$.

(***Digression:*** In quantum mechanics, this angular momentum is quantized, assuming only values that are integer multiples of Planck's reduced constant $\hbar$. This is equivalent to imposing Bohr's condition on the circumference $2\pi r_o$ of the orbit, namely, that the circumference must be an integer-multiple of the particle's DeBroglie wavelength λ, which is related to its momentum via $p=\hbar k=2\pi\hbar/\lambda$. When the particle is in its lowest Bohr orbital, we have $\mathcal{L}=mVr_o=\hbar$.)

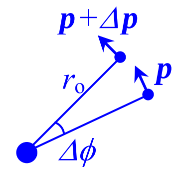
b) Let the charge $-q$ be distributed uniformly around the perimeter of the circle of radius r_o, thus forming a closed loop. The linear charge-density will then be $-q/(2\pi r_o)$, and the loop's current in the $\hat{\boldsymbol{\phi}}$ direction will be $I_o=-qV/(2\pi r_o)$. The magnetic dipole moment is readily seen to be $\boldsymbol{m}=\mu_o(\pi r_o^2)I_o\hat{\mathbf{z}}=-\tfrac{1}{2}\mu_o r_o qV\hat{\mathbf{z}}$. In terms of its orbital angular momentum, we may write the magnetic moment of the circulating negative charge as $\boldsymbol{m}=-(\mu_o q/2m)\boldsymbol{\mathcal{L}}$.

Alternatively, we may find the current I_o by noting that the period of rotation is $T=2\pi r_o/V$. Since, by definition, current is the amount of charge passing through a cross-section of the loop per unit time, we may write $I_o=-q/T=-qV/(2\pi r_o)$. Either way, we find the same answer.

c) The force exerted on the negative charge by the electric field of the central (positive and stationary) charge and by the externally applied, uniform magnetic field is given by the Lorentz law, as follows:

$$\boldsymbol{F} = -q(\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B}) = -q\left[\frac{q\,\hat{\boldsymbol{r}}}{4\pi\varepsilon_o r_o^2} + V\hat{\boldsymbol{\phi}}\times\mu_o H_o\,\hat{\boldsymbol{z}}\right] = -\left[\frac{q^2}{4\pi\,\varepsilon_o r_o^2} + \mu_o q V H_o\right]\hat{\boldsymbol{r}}. \tag{1}$$

d) The linear momentum of the negative charge is $\boldsymbol{p} = mV\hat{\boldsymbol{\phi}}$. In a short time interval Δt, the particle moves a distance $V\Delta t$ along its orbit, sweeping a small angle $\Delta\phi = V\Delta t/r_o$. The momentum $\boldsymbol{p}$ thus rotates through the same angle, corresponding to a change of momentum $\Delta\boldsymbol{p} = -mV\Delta\phi\,\hat{\boldsymbol{r}} = -(mV^2/r_o)\Delta t\,\hat{\boldsymbol{r}}$. Newton's law, $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$, thus yields



$$-\left[\frac{q^2}{4\pi\,\varepsilon_o r_o^2} + \mu_o q V H_o\right]\hat{\boldsymbol{r}} = -\,(mV^2/r_o)\,\hat{\boldsymbol{r}} \quad\rightarrow\quad \frac{q^2}{4\pi\,\varepsilon_o r_o^2} + \mu_o q V H_o = \frac{mV^2}{r_o}. \tag{2}$$

Digression: One may rewrite Eq.(2) as an equation relating the radius r_o to the orbital angular momentum $\mathcal{L} = mVr_o$. Eliminating the particle velocity V from Eq.(2) and rearranging the various terms, we find

$$H_o r_o^{\,2} + \frac{qm}{4\pi\varepsilon_o\mu_o\mathcal{L}}\,r_o - \frac{\mathcal{L}}{\mu_o q} = 0. \tag{3}$$

In the absence of the external field H_o, solving the above equation yields $r_o = 4\pi\varepsilon_o\mathcal{L}^2/(mq^2)$, which, for $\mathcal{L} = \hbar$, is the expression of the Bohr radius for the hydrogen atom. In the presence of an external magnetic field (not too large), the usual assumption is that r_o remains intact while the particle adjusts its velocity V to ensure that Eq.(2) continues to be satisfied. This is equivalent to adjusting the angular momentum $\mathcal{L}$ in order to satisfy Eq.(3) for the constant value of $r_o = 4\pi\varepsilon_o\mathcal{L}_o^2/(mq^2)$. For ordinary magnetic fields, the correction is small, allowing one to replace $\mathcal{L}$ with $(1+\alpha)\mathcal{L}_o$, with the fractional correction-factor α being well below unity. We will have

$$H_o r_o^{\,2} + \frac{qm}{4\pi\varepsilon_o\mu_o(1+\alpha)\mathcal{L}_o}\,r_o - \frac{\mathcal{L}_o(1+\alpha)}{\mu_o q} = 0 \quad\rightarrow\quad H_o r_o^{\,2} + \frac{qm(1-\alpha)}{4\pi\varepsilon_o\mu_o\mathcal{L}_o}\,r_o - \frac{\mathcal{L}_o(1+\alpha)}{\mu_o q} \approx 0$$

$$\rightarrow\quad H_o r_o^{\,2} \approx \left[\frac{qm}{4\pi\varepsilon_o\mu_o\mathcal{L}_o}\,r_o + \frac{\mathcal{L}_o}{\mu_o q}\right]\alpha = \frac{2\alpha\mathcal{L}_o}{\mu_o q} \quad\rightarrow\quad \mathcal{L} = (1+\alpha)\mathcal{L}_o \approx \mathcal{L}_o + \tfrac{1}{2}\mu_o H_o q r_o^{\,2}\hat{\boldsymbol{z}}. \tag{4}$$

The magnetic moment $\underset{\sim}{\boldsymbol{m}}$, being proportional to the angular momentum $\mathcal{L}$, undergoes a similar change in response to the external magnetic field. We will have

$$\underset{\sim}{\boldsymbol{m}} = -(\mu_o q/2m)\,\mathcal{L} \approx \underset{\sim}{\boldsymbol{m}}_o - \frac{(\mu_o q r_o)^2}{4m}H_o\,\hat{\boldsymbol{z}}. \tag{5}$$

This is the classical explanation for the phenomenon of *diamagnetism* associated with bound electrons, as originally formulated by Joseph Larmor. (The conduction electrons' diamagnetism, often referred to as the Landau diamagnetism, has a different explanation.)
