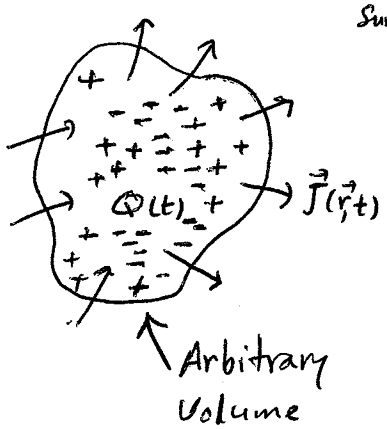
Problem 1) a) $\vec{J}_{free}(\vec{r},t) = \rho_{free}(\vec{r},t)\,\vec{V}(\vec{r},t)$ ✓

b) Units of ρ = Coulomb/m^3; Units of $\vec{V}$ = m/s $\Rightarrow$ Units of $\vec{J} = \frac{\text{Coulomb}}{m^3}\cdot\frac{m}{s} = \frac{\text{Amp.}}{m^2}$ ✓

c) $\vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t) + \frac{\partial}{\partial t}\rho_{free}(\vec{r},t) = 0$ ← differential form ✓

Apply Gauss' Theorem $\Rightarrow \int_{Volume} \vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t)\,d\vec{r} + \int_{Volume} \frac{\partial}{\partial t}\rho_{free}(\vec{r},t)\,d\vec{r} = 0$

$$\Rightarrow \oint_{Surface} \vec{J}_{free}(\vec{r},t)\cdot d\vec{s} + \frac{\partial}{\partial t}Q(t) = 0$$ ← Integral form ✓



The integral of $\vec{J}_{free}(\vec{r},t)$ over the closed surface of the volume of interest yields the net flux of charge out of the volume. (outward flux is because $d\vec{s}$ is defined to point from the inside of the volume to the outside.)

Therefore, during a short time interval Δt, the net charge leaving the volume is equal to $\Delta t \oint_{Surface} \vec{J}(\vec{r},t)\cdot d\vec{s}$. This must be equal to the net drop in the total charge $Q(t)$ contained within the volume. Therefore:

$$\Delta t \oint \vec{J}_{free}(\vec{r},t)\cdot d\vec{s} = -\Delta Q \Rightarrow \oint \vec{J}_{free}(\vec{r},t)\cdot d\vec{s} + \frac{\Delta Q}{\Delta t} = 0$$ ✓

d) $\vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t) + \frac{\partial}{\partial t}\rho_{free}(\vec{r},t) = 0$

Fourier transforming both sides of the above equation yields:

$$\int_{-\infty}^{\infty} \vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t)\,e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{r}\,dt + \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\rho_{free}(\vec{r},t)\,e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\,d\vec{r}\,dt = 0$$

Using integration by parts, each term in the above equation is simplified as follows:

$$\iiiint_{-\infty}^{\infty} \frac{\partial}{\partial x} J_x(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{r}\,dt = \iiint_{-\infty}^{\infty} \Big[\underbrace{J_x(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}}_{\to 0 \text{ as } x\to\pm\infty}\Big]_{-\infty}^{\infty} dy\,dz\,dt$$

$$- \iiiint_{-\infty}^{\infty} J_x(\vec{r},t) \frac{\partial}{\partial x} e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, dx\,dy\,dz\,dt = 0 + i k_x \iiiint_{-\infty}^{\infty} J_x(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{r}\,dt$$

$$= i k_x J_x(\vec{k},\omega) \quad \checkmark$$

Similarly, one can show that

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial y} J_y(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{r}\,dt = i k_y J_y(\vec{k},\omega) \quad \checkmark$$

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial z} J_z(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{r}\,dt = i k_z J_z(\vec{k},\omega) \quad \checkmark$$

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} \rho(\vec{r},t)\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{r}\,dt = -i\omega\, \rho(\vec{k},\omega) \quad \checkmark$$

Therefore, $i k_x J_x(\vec{k},\omega) + i k_y J_y(\vec{k},\omega) + i k_z J_z(\vec{k},\omega) - i\omega\, \rho(\vec{k},\omega) = 0 \Rightarrow$

$$\checkmark \quad \vec{k}\cdot\vec{J}_{free}(\vec{k},\omega) = \omega \rho_{free}(\vec{k},\omega) \quad \leftarrow \text{Continuity equation in the Fourier domain.}$$

Alternative method for part(d):

$$\vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t) + \frac{\partial}{\partial t}\rho_{free}(\vec{r},t) = 0 \Rightarrow \vec{\nabla}\cdot\left\{\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} \vec{J}_{free}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega\right\} +$$

$$\frac{\partial}{\partial t}\left\{\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} \rho_{free}(\vec{k},\omega)\, e^{+i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega\right\} = 0 \Rightarrow$$

$$\int_{-\infty}^{\infty} J_x(\vec{k},\omega) \frac{\partial}{\partial x} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega + \int_{-\infty}^{\infty} J_y(\vec{k},\omega) \frac{\partial}{\partial y} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega + \int_{-\infty}^{\infty} J_z(\vec{k},\omega) \frac{\partial}{\partial z} e^{+i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega$$

$$+ \int_{-\infty}^{\infty} \rho_{free}(\vec{k},\omega) \frac{\partial}{\partial t} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega = 0 \Rightarrow$$

$$i\int_{-\infty}^{\infty} (k_x J_x + k_y J_y + k_z J_z)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega - i\int_{-\infty}^{\infty} \omega \rho_{free}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega = 0 \Rightarrow$$

$$\int_{-\infty}^{\infty} [\vec{k}\cdot\vec{J}_{free}(\vec{k},\omega) - \omega\rho_{free}(\vec{k},\omega)]\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, d\vec{k}\,d\omega = 0 \Rightarrow \vec{k}\cdot\vec{J}_{free}(\vec{k},\omega) - \omega\rho_{free}(\vec{k},\omega) = 0 \quad \checkmark$$

Problem 2) a) $\rho_{total} = \rho_1(\vec{r},t) + \rho_2(\vec{r},t)$ ✓

b) $\vec{J}_{total}(\vec{r},t) = \rho_1(\vec{r},t)\,\vec{V}_1(\vec{r},t) + \rho_2(\vec{r},t)\,\vec{V}_2(\vec{r},t)$ ✓

c) No. The Continuity equation is based on the principle of Conservation of charge. Since individual species 1 & 2 in this example are not Conserved, the Continuity equation does not apply to them as separate entities. For example, the change of charge density with time, $\partial\rho_1/\partial t$ or $\partial\rho_2/\partial t$ may have nothing to do with the influx or outflux of charge produced by the local $\vec{\nabla}\cdot\vec{J}_1(\vec{r},t)$ or $\vec{\nabla}\cdot\vec{J}_2(\vec{r},t)$. Rather, the change of charge density may be caused by recombination of the two species. ✓

d) Yes the Combined system satisfies the Continuity equation, namely,

$$\vec{\nabla}\cdot\vec{J}_{total}(\vec{r},t) + \frac{\partial}{\partial t}\rho_{total}(\vec{r},t) = 0,$$

because the total charge is a conserved entity. When Na^+ and Cl^- ions, for example, Combine, we lose one unit of positive charge and one unit of negative charge, but the total amount of charge in the system remains the same.

Problem 3) a) $\rho_{total}(\vec{r},t) = \rho_{bound}(\vec{r},t) = -\vec{\nabla}\cdot\vec{P}(\vec{r},t)$ ✓

$$\vec{J}_{total}(\vec{r},t) = \vec{J}_{bound}(\vec{r},t) = \frac{\partial}{\partial t}\vec{P}(\vec{r},t)$$ ✓

b) $\vec{J}(\vec{r},t) = \frac{\partial\vec{P}}{\partial t} = -\epsilon_0\chi_0\,\vec{E}(\vec{r})\,\omega_0\sin(\omega_0 t - \phi_0)$

✓ When $\phi_0 = 90^\circ$ we'll have $\vec{J}(\vec{r},t) = \epsilon_0\chi_0\,\vec{E}(\vec{r})\,\omega_0\cos(\omega_0 t)$, which is in phase with $\vec{E}(\vec{r},t)$.

c) $\vec{P}(\vec{r},t) = \text{Real}\{\epsilon_0 \chi_0 e^{i\phi_0} \vec{E}(\vec{r}) e^{-i\omega_0 t}\} = \text{Real}\{\epsilon_0 \chi_0 [\vec{E}'(\vec{r}) + i\vec{E}''(\vec{r})] e^{-i(\omega_0 t - \phi_0)}\}$

$= \epsilon_0 \chi_0 \text{Real}\{[\vec{E}'(\vec{r}) + i\vec{E}''(\vec{r})][\cos(\omega_0 t - \phi_0) - i\sin(\omega_0 t - \phi_0)]\} \Rightarrow$

$$\vec{P}(\vec{r},t) = \epsilon_0 \chi_0 [\vec{E}'(\vec{r})\cos(\omega_0 t - \phi_0) + \vec{E}''(\vec{r})\sin(\omega_0 t - \phi_0)].$$

$$\vec{J}(\vec{r},t) = \frac{\partial \vec{P}}{\partial t} = \epsilon_0 \chi_0 \omega_0 [-\vec{E}'(\vec{r})\sin(\omega_0 t - \phi_0) + \vec{E}''(\vec{r})\cos(\omega_0 t - \phi_0)].$$

$\vec{D}(\vec{r},t) = \epsilon_0 \vec{E}(\vec{r},t) + \vec{P}(\vec{r},t) = \epsilon_0 \text{Real}\{\vec{E}(\vec{r})e^{-i\omega_0 t} + \chi_0 e^{i\phi_0}\vec{E}(\vec{r})e^{-i\omega_0 t}\}$

$= \epsilon_0 \text{Real}\{(1 + \chi_0 e^{i\phi_0})\vec{E}(\vec{r})e^{-i\omega_0 t}\}$

Defining the relative permittivity of the medium $\epsilon e^{i\eta} = 1 + \chi_0 e^{i\phi_0}$, we write

$\vec{D}(\vec{r},t) = \epsilon_0 \text{Real}\{\epsilon \vec{E}(\vec{r})e^{-i(\omega_0 t - \eta)}\} = \epsilon_0 \epsilon [\vec{E}'(\vec{r})\cos(\omega_0 t - \eta) + \vec{E}''(\vec{r})\sin(\omega_0 t - \eta)].$

d) $\vec{\nabla} \cdot \vec{D}(\vec{r},t) = \rho_{free}(\vec{r},t) = 0$ ← Maxwell's first equation, in the absence of ρ_{free}, ensures that $\vec{\nabla} \cdot \vec{D} = 0$.

$\vec{\nabla} \cdot \vec{D}(\vec{r},t) = \epsilon_0 \epsilon \vec{\nabla} \cdot \vec{E}'(\vec{r})\cos(\omega_0 t - \eta) + \epsilon_0 \epsilon \vec{\nabla} \cdot \vec{E}''(\vec{r})\sin(\omega_0 t - \eta) = 0 \Rightarrow$

$\begin{cases} \text{when } \sin(\omega_0 t - \eta) = 0 \text{ we have } \cos(\omega_0 t - \eta) = 1 \Rightarrow \vec{\nabla} \cdot \vec{E}'(\vec{r}) = 0 \\ \text{Similarly, when } \cos(\omega_0 t - \eta) = 0 \Rightarrow \sin(\omega_0 t - \eta) = 1 \Rightarrow \nabla \cdot \vec{E}''(\vec{r}) = 0 \end{cases}$

Consequently, $\vec{\nabla} \cdot \vec{E}(\vec{r},t) = 0$ and $\vec{\nabla} \cdot \vec{P}(\vec{r},t) = 0$. ✓

Note that $\vec{\nabla} \cdot \vec{P} = 0$ implies that within a homogeneous, linear, isotropic medium the density of bound charges is zero, namely, $\rho_{bound}(\vec{r},t) = -\vec{\nabla} \cdot \vec{P}(\vec{r},t) = 0$.

Problem 4) a) $\vec{M}(\vec{r},t) = \frac{M_0 R_1}{r}\left[\mathrm{Circ}(r/R_2) - \mathrm{Circ}(r/R_1)\right]\mathrm{Rect}(z/h)\,\hat{r}.$

b) $\vec{J}_b^{(e)}(\vec{r},t) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{M}(\vec{r},t) = \frac{1}{\mu_0}\frac{\partial M_r}{\partial z}\hat{\phi} = \frac{M_0 R_1}{\mu_0 r}\left[\mathrm{Circ}(r/R_2) - \mathrm{Circ}(r/R_1)\right]\left[\delta(z+\frac{h}{2}) - \delta(z-\frac{h}{2})\right]\hat{\phi}$

$$\rho_b^{(m)}(\vec{r},t) = -\vec{\nabla}\cdot\vec{M}(\vec{r},t) = \frac{-1}{r}\frac{\partial}{\partial r}(rM_r) = \frac{M_0 R_1}{r}\left[\delta(r-R_2) - \delta(r-R_1)\right]\mathrm{Rect}(z/h) \Longrightarrow$$

$$\rho_b^{(m)}(\vec{r},t) = M_0\left[\frac{R_1}{R_2}\delta(r-R_2) - \delta(r-R_1)\right]\mathrm{Rect}(z/h).$$

c) On the inner and outer cylidrical surfaces the velocity is $R_1\omega_0$ and $R_2\omega_0$, respectively. Since $\vec{J}_b^{(m)}(\vec{r},t) = \rho_b^{(m)}(\vec{r},t)\,\vec{V}(\vec{r},t)$, if we write $\vec{V}(\vec{r},t) = r\omega_0\,\hat{\phi}$, the delta functions present in the expression of $\rho_b^{(m)}$ will ensure that the inner cylinder charges are multiplied with $R_1\omega_0$, while the outer cylinder charges are multiplied with $R_2\omega_0$. We will have:

$$\vec{J}_b^{(m)}(\vec{r},t) = \rho_b^{(m)}(\vec{r},t)\,r\omega_0\hat{\phi} = M_0\omega_0\left[R_1\delta(r-R_2) - R_1\delta(r-R_1)\right]\mathrm{Rect}(z/h)\,\hat{\phi} \Rightarrow$$

$$\vec{J}_b^{(m)}(\vec{r},t) = M_0 R_1\omega_0\left[\delta(r-R_2) - \delta(r-R_1)\right]\mathrm{Rect}(z/h)\,\hat{\phi}.$$

d) Maxwell's 3rd equation: $\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t) \Rightarrow \vec{\nabla}\times\epsilon_0\vec{E}(\vec{r},t) = -\epsilon_0\frac{\partial\vec{M}}{\partial t} - \mu_0\epsilon_0\frac{\partial\vec{H}}{\partial t}$

$$\Rightarrow \vec{\nabla}\times\vec{D}(\vec{r},t) = -\epsilon_0\left(\frac{\partial\vec{M}}{\partial t} - \frac{1}{\epsilon_0}\vec{\nabla}\times\vec{P}\right) - \mu_0\epsilon_0\frac{\partial\vec{H}(\vec{r},t)}{\partial t}$$

Comparison with Maxwell's 2nd equation, $\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_{free} + \frac{\partial\vec{P}}{\partial t} + \frac{1}{\mu_0}\vec{\nabla}\times\vec{M}\right) + \mu_0\epsilon_0\frac{\partial\vec{E}(\vec{r},t)}{\partial t}$, shows that the bound <u>magnetic</u> current may be defined as $\vec{J}_b^{(m)}(\vec{r},t) = \frac{\partial\vec{M}}{\partial t} - \frac{1}{\epsilon_0}\vec{\nabla}\times\vec{P}$.

In the present problem $\partial\vec{M}/\partial t = 0$, but $\vec{J}_b^{(m)}$ is associated with $\vec{\nabla}\times\vec{P}$, as follows:

$$-\frac{1}{\epsilon_0}\vec{\nabla}\times\vec{P}(\vec{r},t) = -\frac{1}{\epsilon_0}\vec{\nabla}\times\left\{P_0\left[\mathrm{Circ}(r/R_2) - \mathrm{Circ}(r/R_1)\right]\mathrm{Rect}(z/h)\,\hat{z}\right\} = \frac{P_0}{\epsilon_0}\frac{\partial}{\partial r}\left[\mathrm{Circ}(r/R_2) - \mathrm{Circ}(r/R_1)\right]\mathrm{Rect}(z/h)\,\hat{\phi}$$

$$= (P_0/\epsilon_0)\left[-\delta(r-R_2) + \delta(r-R_1)\right]\mathrm{Rect}(z/h)\,\hat{\phi}$$

Comparison with $\vec{J}_b^{(m)}(\vec{r},t)$ in part (c) reveals that $P_0 = -\epsilon_0 M_0 R_1\omega_0$.