1) a) There is no dependence on ϕ, thus ψ is a function of ρ and θ only. Moreover, there is symmetry between points above and below the XY-plane, having the same x and y coordinates, but differing signs for the z-coordinate, namely, $\pm z$. Thus $\psi(\rho,\theta) = \psi(\rho,\pi-\theta)$.

b) $\vec{E} = -\vec{\nabla}\psi \Rightarrow \vec{E}(\rho,\theta,\phi) = -\frac{\partial\psi}{\partial\rho}\hat{\rho} - \frac{1}{\rho}\frac{\partial\psi}{\partial\theta}\hat{\theta} - \frac{1}{\rho\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\phi}$.

Since ψ is not dependent on ϕ, we conclude that $E_\phi = 0$. Thus, $E_\rho(\rho,\theta) = -\frac{\partial}{\partial\rho}\psi(\rho,\theta)$ and $E_\theta(\rho,\theta) = -\frac{1}{\rho}\frac{\partial}{\partial\theta}\psi(\rho,\theta)$. Symmetry with respect to the xy-plane shows that $E_\rho(\rho,\theta) = E_\rho(\rho,\pi-\theta)$ and $E_\theta(\rho,\theta) = -E_\theta(\rho,\pi-\theta)$.

c) In general, we have already seen that $E_\phi = 0$ everywhere. As for E_θ, the odd symmetry obtained in part (b), namely, $E_\theta(\rho,\theta) = -E_\theta(\rho,\pi-\theta)$, dictates that at $\theta=\pi/2$ we must have $E_\theta(\rho,\pi/2) = -E_\theta(\rho,\pi/2)$. This is possible only if $E_\theta(\rho,\pi/2)=0$. We conclude, therefore, that at $(\rho>R, \theta=\pi/2, \phi)$ $E_\theta = E_\phi = 0$.

d) Using Gauss' law, $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$, applied to a small pillbox at the surface of the disk, and the symmetry condition that $E_\theta^{\text{above}} = -E_\theta^{\text{below}}$ we find $E_\theta^{(A)} = -E_\theta^{(B)} = -\frac{1}{2}\sigma_s/\epsilon_0$.

e) We can only say that $E_\rho^{(A)} = E_\rho^{(B)}$, either from symmetry discussed in part (b), or from the boundary condition derived from $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ that asserts that the tangential component of $\vec{E}$ is always continuous.

Note that the discontinuity of $E_\theta(\rho<R, \theta=\frac{\pi}{2}, \phi)$ at the disk surface does <u>not</u> imply a discontinuity for $\psi(\rho,\theta)$ at the disk surface, but only for its first derivative.

2) a) This is <u>both</u> an electrostatic problem (because σ_s is <u>not</u> a function of time) and a magneto-static problem (because J_s is <u>not</u> a function of time, and no charge is accumulating anywhere).

Since the charge density distribution σ_s is the same as that in Problem 1, the scalar potential ψ is the same. Moreover, $\vec{E} = -\vec{\nabla}\psi - \partial\vec{A}/\partial t$, but in magneto-statics $\vec{A}$ is <u>not</u> a function of time, hence $\vec{E} = -\vec{\nabla}\psi$ and, therefore, $\vec{E}$ is also the same as in Problem 1.

b) The linear velocity of the disk at a radius ρ is $\vec{V} = \rho\omega\hat{\phi}$.
Therefore, $\vec{J}_s(\rho) = \sigma_s\vec{V} = \sigma_s\rho\omega\hat{\phi}$, $(\rho \le R)$.

c) In cylindrical coordinates $\vec{\nabla}\cdot\vec{J} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho \cancelto{0}{J_\rho}) + \frac{1}{\rho}\frac{\partial}{\partial\phi}J_\phi + \frac{\partial}{\partial z}\cancelto{0}{J_z} =$
$\frac{1}{\rho}\frac{\partial}{\partial\phi}J_\phi = \frac{1}{\rho}\frac{\partial}{\partial\phi}(\sigma_s\rho\omega) = 0$.

Also, σ_s is time-independent; therefore $\frac{\partial\sigma_s}{\partial t} = 0$. $\Rightarrow \vec{\nabla}\cdot\vec{J}_s + \frac{\partial\sigma_s}{\partial t} = 0$ ✓

d) First, $\vec{A}(\rho,\phi,z)$ does <u>not</u> depend on ϕ. Second, $\vec{A}(\rho,z)$ does <u>not</u> have a z-component, because $\vec{J}_s$ does not have a z-component. Third, $\vec{A}$ does <u>not</u> have a ρ-component, because for each point on the disk that contributes an A_ρ at an observation point, there is always another point on the disk which cancels that contribution to A_ρ.
We conclude that the vector potential field is $A_\phi(\rho,z)\hat{\phi}$.
Moreover, on the z-axis the vector potential is zero, i.e., $A_\phi(0,z)=0$.
Finally, there is symmetry with respect to the xy-plane, namely,
$A_\phi(\rho,z) = A_\phi(\rho,-z)$.

e) $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \mu_0 \vec{H} = \left(\frac{1}{\rho}\frac{\partial \cancel{A_z}}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial \cancel{A_\rho}}{\partial z} - \frac{\partial \cancel{A_z}}{\partial \rho}\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial \cancel{A_\rho}}{\partial \phi}\right)\hat{z}$

$\Rightarrow \mu_0 \vec{H} = -\frac{\partial A_\phi}{\partial z}\hat{\rho} + \left(\frac{A_\phi}{\rho} + \frac{\partial A_\phi}{\partial \rho}\right)\hat{z}.$

✓ Thus $H_\phi = 0$, $\mu_0 H_\rho(\rho, z) = -\frac{\partial A_\phi}{\partial z}$, and $\mu_0 H_z(\rho, z) = \frac{A_\phi(\rho, z)}{\rho} + \frac{\partial}{\partial \rho} A_\phi(\rho, z)$.

✓ On the z-axis, $H_\rho = 0$ because $A_\phi(0, z) = 0$.

✓ $H_\rho(\rho, z) = -H_\rho(\rho, -z)$ because of the symmetry of A_ϕ with respect to the xy-plane.

✓ $H_z(\rho, z) = H_z(\rho, -z)$, again because of the symmetry of A_ϕ w.r.t. xy-plane.

f) First, $H_\phi = 0$ everywhere. Second, $H_\rho(\rho, z) = -H_\rho(\rho, -z)$ implies that at $z=0$ we must have $H_\rho(\rho, 0) = -H_\rho(\rho, 0)$, which is possible only if $H_\rho(\rho, 0) = 0$. Therefore, at $(\rho > R, \phi, z=0)$ we have $H_\phi = H_\rho = 0$.

g) Applying Ampere's law, $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$, to a small rectangular path that crosses the disk, we find that $H_\rho^{(A)} - H_\rho^{(B)} = J_s = \sigma_s \rho \omega \Rightarrow$
$H_\rho^{(A)} = -H_\rho^{(B)} = \frac{1}{2}\sigma_s \rho \omega.$

h) We can only say that $H_z^{(A)} = H_z^{(B)}$ from the symmetry discussed in part (e), or from the boundary condition derived from $\vec{\nabla} \cdot \vec{B} = 0$ that asserts that the perpendicular component of $\vec{B}$ is always continuous.

i) For a ring of radius ρ and width $\Delta\rho$, the current I is $J_s \Delta\rho = \sigma_s \rho \omega \Delta\rho$, and the loop area is $\pi\rho^2$. Therefore, $\Delta\vec{m} = (\pi\rho^2)(\sigma_s \rho\omega)\Delta\rho\,\hat{z}$.

Consequently $\vec{m} = \int_{\rho=0}^{R} \pi\sigma_s\omega\rho^3\hat{z}\,d\rho = \frac{1}{4}\pi\sigma_s\omega R^4\hat{z}$. This definition of $\vec{m}$ is consistent with $\vec{B} = \mu_0(\vec{H} + \vec{m})$. It must be multiplied by μ_0 if $\vec{B} = \mu_0\vec{H} + \vec{m}$.

3) a) The current starts flowing at $t=0$, and the light bulb turns on. ~~Ra current~~. The flow of current in the metallic rod produces a braking force on the rod (as a result of interaction between the current and the B-field). The rod, therefore, slows down, thus reducing the current $I(t)$. The light bulb gets dimmer and the rod moves more slowly as time goes on. Eventually, the rod stops and the light bulb turns off.

b) $\vec{F} = q\vec{V}\times\vec{B}$ and $\vec{F} = q\vec{E}_{eff} \Rightarrow \vec{E}_{eff} = \vec{V}\times\vec{B} \Rightarrow E_{eff} = B_0 v(t) \Rightarrow$

Induced Voltage $V(t) = \int\limits_{\substack{\text{length}\\ \text{of rod}}} \vec{E}_{eff}\cdot d\vec{\ell} = B_0 L v(t)$ ✓

$I(t) = \frac{V(t)}{R} = (B_0 L/R)\, v(t)$ ✓

c) Conduction charge density in the rod $= \rho$; rod cross-sectional area $= S$; charge velocity within the rod (along its length) $= \vec{V}$; current density $\vec{J} = \rho\vec{V}$; current $I = \vec{J}\cdot\vec{S} = \rho S V$; volume of the rod $= LS$; total conduction charge within the volume of the rod $= \rho L S$; force on the conduction current $= (\rho L S)\vec{V}\times\vec{B} = -\rho L S V B_0 = -L B_0 I(t)$. The minus sign indicates that this force is opposite the direction of motion of the rod.

$$\vec{F}(t) = M\frac{d\vec{v}(t)}{dt} \Rightarrow -LB_0 I(t) = M\frac{dv(t)}{dt} \Rightarrow -\frac{L^2B_0^2}{R}v(t) = M\frac{dv(t)}{dt} \Rightarrow$$

$$\frac{dv(t)}{v(t)} = -\frac{L^2B_0^2}{MR}dt \Rightarrow \int_0^t \frac{dv(t)}{v(t)} = -\left(\frac{L^2B_0^2}{MR}\right)\int_0^t dt \Rightarrow \ln\left[\frac{v(t)}{v_0}\right] = -\left(\frac{L^2B_0^2}{MR}\right)t$$

$$\Rightarrow v(t) = v_0 \exp\left[-\left(\frac{L^2B_0^2}{MR}\right)t\right]$$

d) $\text{Energy} = \int_0^\infty V(t)I(t)\,dt = (B_0^2L^2/R)\int_0^\infty v^2(t)\,dt = (B_0^2L^2v_0^2/R)\int_0^\infty \exp\left[-2\left(\frac{L^2B_0^2}{MR}\right)t\right]dt$

$= (B_0^2L^2v_0^2/R)\,/\,[2(L^2B_0^2/MR)] = \frac{1}{2}Mv_0^2$ ✓