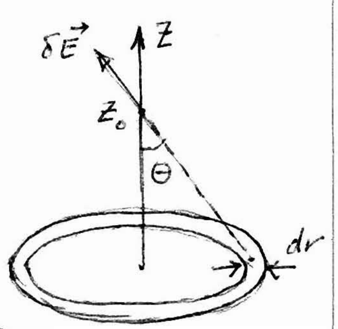
Problem 1) a) $E_z(z=z_0) = \frac{1}{4\pi\epsilon_0}\int_{r=0}^{R} \frac{2\pi r\sigma_s \cos\theta}{r^2+z_0^2}\,dr$



$$= \frac{1}{4\pi\epsilon_0}\int_{r=0}^{R}\frac{2\pi r\sigma_s z_0}{(r^2+z_0^2)^{3/2}}\,dr = \frac{\sigma_s}{2\epsilon_0}\int_{x=0}^{R/z_0}\frac{x\,dx}{(1+x^2)^{3/2}}$$

$$= -\frac{\sigma_s}{2\epsilon_0}(1+x^2)^{-1/2}\Big|_{x=0}^{R/z_0} \Rightarrow E_z(z=z_0) = \frac{\sigma_s}{2\epsilon_0}\left(1-\frac{1}{\sqrt{1+\frac{R^2}{z_0^2}}}\right)$$

b) $$\psi(z=z_0) = \frac{1}{4\pi\epsilon_0}\int_{r=0}^{R}\frac{2\pi r\sigma_s}{\sqrt{r^2+z_0^2}}\,dr = \frac{\sigma_s z_0}{2\epsilon_0}\int_{x=0}^{R/z_0}\frac{x\,dx}{\sqrt{1+x^2}} = \frac{\sigma_s z_0}{2\epsilon_0}\sqrt{1+x^2}\Big|_{x=0}^{R/z_0}$$

$$\Rightarrow \psi(z=z_0) = \frac{\sigma_s}{2\epsilon_0}\left(\sqrt{R^2+z_0^2}-z_0\right)$$

c) $$E_z(z=z_0) = -\frac{\partial\psi}{\partial z}\Big|_{z=z_0} = -\frac{\sigma_s}{2\epsilon_0}\left(\frac{z_0}{\sqrt{R^2+z_0^2}}-1\right) = \frac{\sigma_s}{2\epsilon_0}\left(1-\frac{1}{\sqrt{1+\frac{R^2}{z_0^2}}}\right)$$ ✓

d) When $R\to\infty$ the second term in the expression for E_z approaches zero. We then have $E_z(z=z_0) \xrightarrow[R\to\infty]{} \frac{\sigma_s}{2\epsilon_0}$.

e) $$\frac{\sigma_s}{2\epsilon_0}\left(1-\frac{1}{\sqrt{1+R^2/z_0^2}}\right) \geq 0.99\frac{\sigma_s}{2\epsilon_0} \Rightarrow \frac{1}{\sqrt{1+R^2/z_0^2}} \leq 0.01 \Rightarrow 1+\frac{R^2}{z_0^2} \geq 10{,}000 \Rightarrow$$

$$R \geq \sqrt{9999}\,z_0 \Rightarrow R \gtrsim 100 z_0$$

f) $$Q = \pi R^2\sigma_s \Rightarrow E_z(z=z_0) = \frac{Q}{2\pi R^2\epsilon_0}\left(1-\frac{1}{\sqrt{1+R^2/z_0^2}}\right) = \frac{Q}{2\pi\epsilon_0 R^2}\left[1-\left(1+\frac{R^2}{z_0^2}\right)^{-1/2}\right]$$

$$\simeq \frac{Q}{2\pi\epsilon_0 \cancel{R^2}}\left(\cancel{1}-\cancel{1}+\frac{1}{2}\frac{\cancel{R^2}}{z_0^2}\right) = \frac{Q}{4\pi\epsilon_0 z_0^2}$$

← This is the field of a point charge Q at a distance z_0.

Problem 2) a) $\vec{H}(t) = J_{S_0} \sin(2\pi f t)\,\hat{z}$ ← Same as surface current density, but at 90°. (Use the right-hand rule.)

$\vec{B}(t) = \mu_0 \vec{H}(t) = \mu_0 J_{S_0} \sin(2\pi f t)\,\hat{z}$

Outside the solenoid the $\vec{H}$ and $\vec{B}$ fields are approximately zero, because the frequency f is assumed to be small.

b) Total magnetic flux $\Phi = \pi R_1^2 B(t) = \mu_0 \pi R_1^2 J_{S_0} \sin(2\pi f t)$

Faraday's law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \oint_{loop} \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t}\Phi$

The Voltage $V(t)$ is defined in the figure such that $\oint \vec{E} \cdot d\vec{\ell}$ is taken in the clockwise direction (when seen from above). This introduces another minus sign into the equation. Consequently,

$$V(t) = -\oint \vec{E} \cdot d\vec{\ell} = \frac{\partial}{\partial t}\Phi = 2\pi^2 \mu_0 f R_1^2 J_{S_0} \cos(2\pi f t)$$

c) In Assignment #3, Problem 6, we saw that the vector potential $\vec{A}(\vec{r})$ of an infinitely long solenoid at a radius $r > R_1$ is given by:

$$\vec{A}(\vec{r}) = \frac{1}{2}(\mu_0 J_s R_1^2 / r)\hat{\phi}$$

In the present quasi-static problem, $J_s = J_{S_0} \sin(2\pi f t)$ and $r = R_2$.

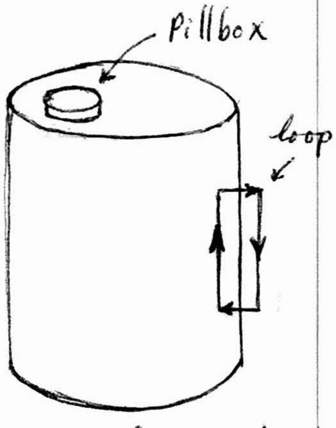
Therefore, $\vec{A}(\vec{r}) = \frac{1}{2}\mu_0 (R_1^2/R_2) J_{S_0} \sin(2\pi f t)\,\hat{\phi}$.

Since $\vec{\nabla} \cdot \vec{J} = 0$ for the surface current running around the solenoid there are no free charges in this problem (i.e., ρ or $\sigma_s = 0$). The scalar potential φ is therefore zero. Consequently,

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} = -\frac{\partial \vec{A}}{\partial t} = -\pi \mu_0 f (R_1^2/R_2) J_{S_0} \cos(2\pi f t)\,\hat{\phi} \leftarrow \text{at } r = R_2$$

$$V(t) = -\oint \vec{E} \cdot d\vec{\ell} = -2\pi R_2 E = 2\pi^2 \mu_0 f R_1^2 J_{S_0} \cos(2\pi f t). \checkmark$$

Problem 3) a) A thin pillbox can be used at the top and bottom surfaces of the cylinder to determine $\vec{\nabla}\cdot\vec{M}$. Let the pillbox have surface area S and thickness τ. Then $\int_{\text{Pillbox}} \vec{M}\cdot d\vec{s} = -M_0 S$.



Dividing by the volume $S\tau$, we find $\vec{\nabla}\cdot\vec{M} = -M_0/\tau$.
Thus, at the top of the cylinder, $\rho_m = -\vec{\nabla}\cdot\vec{M} = M_0/\tau$. Since τ can be arbitrarily small, we use $\sigma_{sm} = \rho_m \tau$ instead. Therefore, at the top, $\sigma_{sm} = M_0$.
Similarly, at the bottom, $\sigma_{sm} = -M_0$.

To find $\vec{\nabla}\times\vec{M}$, we use a narrow rectangular loop at the cylinder's sidewall. Again the width w of this loop must approach zero. The integral around the loop is $\oint \vec{M}\cdot d\vec{\ell} = M_0 \ell$, where ℓ is the length of the rectangle. Dividing by the loop area, ℓw, yields $\vec{\nabla}\times\vec{M} = (M_0/w)\hat{\phi}$. Since w can be arbitrarily small, we'll use $\vec{J}_{sm} = \vec{J}_m w$ instead. Consequently, $\vec{J}_{sm} = M_0\hat{\phi}$ on the cylinder's sidewall.

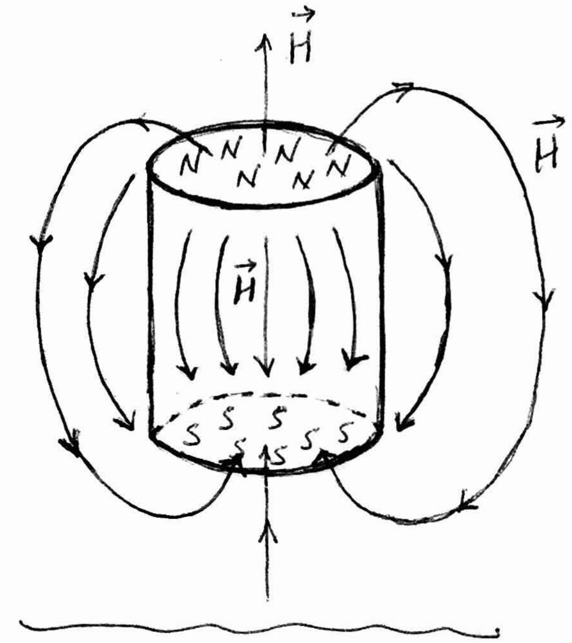
b) $\vec{B} = \mu_0(\vec{H}+\vec{M})$ and $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\nabla}\cdot(\vec{H}+\vec{M}) = 0 \Rightarrow \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} \Rightarrow$
$\vec{\nabla}\cdot\vec{H} = \rho_m$. This is similar to the first Maxwell equation, $\vec{\nabla}\cdot\vec{D} = \rho_e$, where ρ_e is the density of free electric charges. Here $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ is the displacement field. In the absence of $\vec{P}$, we'll have $\vec{D} = \epsilon_0\vec{E}$.
In electrostatics, $\vec{\nabla}\times\vec{E} = 0$, which, in the absence of $\vec{P}$, yields: $\vec{\nabla}\times\vec{D} = 0$.
Thus, in the absence of $\vec{P}$, the equations that determine $\vec{D}$ are $\vec{\nabla}\cdot\vec{D} = \rho_e$ and $\vec{\nabla}\times\vec{D} = 0$ in the electrostatic regime.

In magnetostatics, when there is a distribution of magnetization, $\vec{M}(\vec{r})$, but no free currents (i.e., $\vec{J}_{\text{free}} = 0$), we'll have $\vec{\nabla}\cdot\vec{H} = \rho_m$ and $\vec{\nabla}\times\vec{H} = 0$.

The situation, therefore, is similar to that in electrostatics for $\vec{D}$. The magnetic charge density ρ_m (or σ_{ms}) gives rise to the $\vec{H}$-field in exactly the same way that the electric charge density ρ_e (or σ_{es}) would give rise to the D-field. The lines of $\vec{H}$ for the cylindrical permanent magnet are shown below.

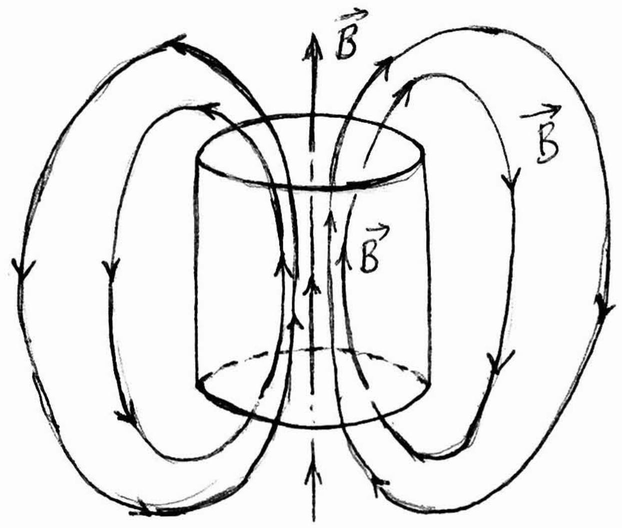
✓ Charge density at the top (north pole) $= M_0$

✓ Charge density at the bottom (south pole) $= -M_0$



c) The magnetization $\vec{M}(\vec{r})$ is produced by the magnetic current distribution $\vec{J}_m(\vec{r}) = \vec{\nabla} \times \vec{M}(\vec{r})$. This current produces a B-field in accordance with the Biot–Savart law, namely, $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}_m(\vec{r}\,') \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} dv'$.

In the case of the cylindrical permanent magnet, the current $\vec{J}_m$ is a surface current, $\vec{J}_{ms}(\vec{r}) = M_0 \hat{\phi}$ on the cylinder's sidewall. The situation, therefore, is the same as a solenoid of length L and radius R, with a uniform surface current $\vec{J}_{ms} = M_0 \hat{\phi}$. The B-field profile of this permanent magnet is shown in the figure →



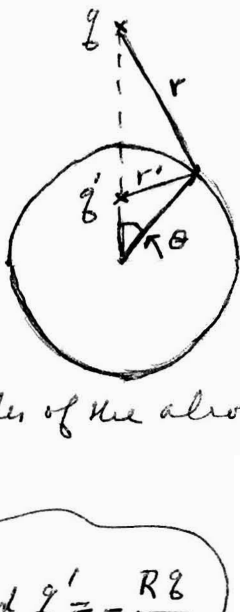
Inside the magnet $\vec{B}=\mu_0(\vec{H}+\vec{M})$, with $\vec{M}=M_0\hat{z}$. Outside the magnet, $\vec{B}=\mu_0\vec{H}$.

In general, it is possible to prove that, for a static magnetization distribution $\vec{M}(\vec{r})$, the magnetic charge density $\rho_m=-\vec{\nabla}\cdot\vec{M}$ produces the scalar magnetic potential $\psi(\vec{r})=\frac{1}{4\pi}\int\frac{\rho_m(\vec{r}')}{|\vec{r}-\vec{r}'|}dv'$, from which one obtains the magnetic H-field as follows: $\vec{H}(\vec{r})=-\vec{\nabla}\psi=\frac{1}{4\pi}\int(\vec{\nabla}\cdot\vec{M})\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)dv'$. Similarly, the magnetic current density $\vec{J}_m=\vec{\nabla}\times\vec{M}$ produces the B-field via the Biot-Savart law as follows: $\vec{B}(\vec{r})=\frac{-\mu_0}{4\pi}\int(\vec{\nabla}\times\vec{M})\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)dv'$. (The minus sign is due to the fact that $\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)=-\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$.) One can show that, in general,

$$\text{(I)}\qquad \int_{\text{all space}}\left[(\vec{\nabla}\cdot\vec{M})\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)+(\vec{\nabla}\times\vec{M})\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]dv'=-4\pi\vec{M}(\vec{r}),$$

which proves that the $\vec{H}$ and $\vec{B}$ thus calculated for a given $\vec{M}(\vec{r})$ satisfy the relation $\vec{B}(\vec{r})=\mu_0\left[\vec{H}(\vec{r})+\vec{M}(\vec{r})\right]$. One way to prove Eq. (I) above is by Fourier transforming the equation and demonstrating its validity in the $\vec{k}$-space.

Problem 4) a) $\psi(r=R,\theta)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r}+\frac{q'}{r'}\right)=0$



$$\Rightarrow\ \frac{q}{r}+\frac{q'}{r'}=0\ \Rightarrow\ \frac{q}{q'}=-\frac{\sqrt{R^2+d^2-2Rd\cos\theta}}{\sqrt{R^2+d'^2-2Rd'\cos\theta}}\quad\text{for } 0\le\theta\le\pi.$$

$$\Rightarrow\ q^2(R^2+d'^2-2Rd'\cos\theta)=q'^2(R^2+d^2-2Rd\cos\theta)$$

Since the above equation must be valid for all θ in the interval $[0,\pi]$, the coefficients of $\cos\theta$ on both sides of the above equation must be equal. Consequently,

$$\begin{cases}2Rd'q^2=2Rdq'^2\\ q^2(R^2+d'^2)=q'^2(R^2+d^2)\end{cases}\Rightarrow\begin{cases}q^2/q'^2=d/d'\\ R^2=dd'\end{cases}\Rightarrow d'=\frac{R^2}{d}\ \text{ and }\ q'=-\frac{Rq}{d}$$

b) The image charge q', together with the charge q, produces the same potential at the sphere's surface (i.e., $\psi=0$). Therefore, the E-field in the region outside the sphere must be the same, whether the sphere is held at zero potential, or removed and replaced with the charge q' at distance d' from the center. Now, if a Gaussian surface is drawn immediately outside the sphere's surface, the flux of $\epsilon_0 \vec{E}$ on its surface must equal the charge on the sphere's surface. The flux of $\epsilon_0 \vec{E}$, however, is equal to q' when the sphere is replaced with the image charge. Therefore, the total charge on the sphere is $q' = -\frac{Rq}{d}$.

c) q'' must be placed at the center of the sphere in order to produce a constant potential $\psi_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{R}$ at the surface of the sphere. Therefore, $q'' = 4\pi\epsilon_0 R \psi_0$.

d) For exactly the same reasons stated in Part (b), the total charge collected on the sphere's surface will be $q' + q''$.

e) The surface charge density σ_s at each point on a conductor's surface is equal to $\epsilon_0 E_\perp$ where $E_\perp$ is the perpendicular component of the E-field immediately outside the conductor. ($E_\parallel$ is always zero for a perfect conductor.) The E-field of q and q' at the spherical surface must then be calculated in order to determine the corresponding σ_s'. For the charge q'', the surface charge density is uniform, given by $\sigma_s'' = \frac{q''}{4\pi R^2}$. This charge density must then be added to σ_s' (corresponding to the case of zero potential) to yield the total surface charge density σ_s. In the following we calculate the charge density due to q and q' only (i.e., case of $\psi_0 = 0$). The easiest way to determine $E_\perp$ at the surface is by way of computing $-\vec{\nabla}\psi = -\frac{\partial \psi}{\partial n}$.

where $\vec{n}$ is the normal to the sphere surface. From part (a) we write the potential $\Psi(R,\theta)$ at the sphere's surface as follows:

$$\Psi(R,\theta)=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{R^2+d^2-2Rd\cos\theta}}+\frac{q'}{\sqrt{R^2+d'^2-2Rd'\cos\theta}}\right)$$

Assuming that q, q', d, d' and θ are fixed, a small change in R will cause a small change in Ψ, with $E_\perp=-\frac{\partial\Psi}{\partial n}=-\frac{\partial\Psi(R,\theta)}{\partial R}$. Therefore,

$$E_\perp=\frac{1}{4\pi\epsilon_0}\left[\frac{q(R-d\cos\theta)}{(R^2+d^2-2Rd\cos\theta)^{3/2}}+\frac{q'(R-d'\cos\theta)}{(R^2+d'^2-2Rd'\cos\theta)^{3/2}}\right].$$

Next we set $R=\sqrt{dd'}$, $q'=-Rq/d$, and use the fact that $\sigma_s(R,\theta)=\epsilon_0 E_\perp$.

$$\sigma_s(R,\theta)=\epsilon_0 E_\perp(R,\theta)=\frac{1}{4\pi}\frac{q}{(d+d'-2R\cos\theta)^{3/2}}\left[\frac{R-d\cos\theta}{d^{3/2}}-\frac{(R/d)(R-d'\cos\theta)}{d'^{3/2}}\right]$$

$$=\frac{q}{4\pi d(d+d'-2R\cos\theta)^{3/2}}\left[\sqrt{d'}-\cancel{\sqrt{d}\cos\theta}-\frac{d}{\sqrt{d'}}+\cancel{\sqrt{d}\cos\theta}\right]\Rightarrow$$

$$\sigma_s(R,\theta)=\frac{q(d'-d)}{4\pi d\sqrt{d'}(d+d'-2R\cos\theta)^{3/2}}=\frac{q(R^2-d^2)}{4\pi R(R^2+d^2-2Rd\cos\theta)^{3/2}}$$

The total charge accumulated on the sphere surface is thus given by

$$\text{Total charge}=\int_{\theta=0}^{\pi}2\pi R^2\sin\theta\,\sigma_s(R,\theta)\,d\theta=\frac{qR^2(R^2-d^2)}{2R}\int_0^{\pi}\frac{\sin\theta}{(R^2+d^2-2Rd\cos\theta)^{3/2}}d\theta$$

$$=-\frac{q(R^2-d^2)}{2d}\frac{1}{\sqrt{R^2+d^2-2Rd\cos\theta}}\Bigg|_0^{\pi}=\frac{q(d^2-R^2)}{2d}\left(\frac{1}{d+R}-\frac{1}{d-R}\right)$$

$$\Rightarrow \text{Total charge}=-\frac{qR}{d}=q'$$