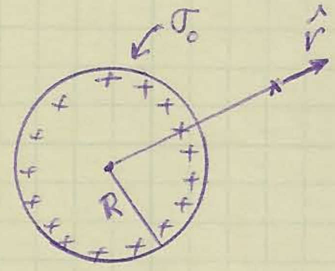
1) a) Total charge $Q = 4\pi R^2 \sigma_0$



Symmetry dictates that the field $\vec{E}$ be radial (i.e., aligned with $\hat{r}$), and also have the same magnitude on a sphere of radius r.

$$\oint_{\substack{\text{Surface of}\\ \text{Sphere of}\\ \text{radius } r}} \vec{E}\cdot d\vec{s} = Q/\epsilon_0 \Rightarrow 4\pi r^2 E_r(\vec{r}) = \begin{cases} 4\pi R^2\sigma_0/\epsilon_0 & \text{if } r>R \\ 0 & \text{if } r<R \end{cases}$$

Thus $$\vec{E}(\vec{r}) = E_r(\vec{r})\,\hat{r} = \begin{cases} R^2\sigma_0/(\epsilon_0 r^2) & r>R \\ 0 & r<R \end{cases}$$

b) $$\text{E-field's total energy} = \int \tfrac{1}{2}\epsilon_0 |\vec{E}|^2 dv = \frac{1}{2}\epsilon_0 \int_{r=R}^{\infty} 4\pi r^2 \left(\frac{R^2\sigma_0}{\epsilon_0 r^2}\right)^2 dr$$

$$= \frac{2\pi R^4\sigma_0^2}{\epsilon_0}\int_{r=R}^{\infty}\frac{dr}{r^2} = \frac{2\pi R^3\sigma_0^2}{\epsilon_0}$$

c) The effective $\vec{E}$-field acting on the surface charges is the average of the $\vec{E}$-field just below and just above the sphere's surface. Thus $\vec{E}_{eff} = \frac{1}{2}\frac{\sigma_0}{\epsilon_0}\hat{r}.$ The force acting on the surface charges is $\vec{F} = \sigma_0 \vec{E}_{eff} = \frac{1}{2}\frac{\sigma_0^2}{\epsilon_0}\hat{r}$ per unit area. When the shell radius shrinks from R to R$-\Delta$R, work must be done against this force, which is trying to expand the shell.

Total work $= 4\pi R^2 \sigma_0 \vec{E}_{eff}\cdot(\Delta R\,\hat{r}) = 2\pi R^2\sigma_0^2 \Delta R/\epsilon_0$

d) We express the field's energy and the work done to shrink the shell in terms of the total charge Q of the spherical shell, because Q is independent of the radius R, whereas σ_0 will vary if R is changed. Thus: Total field energy $= \dfrac{Q^2}{8\pi\epsilon_0 R}$, Total work $= \dfrac{Q^2\Delta R}{8\pi\epsilon_0 R^2}$

$$\frac{d}{dR}(\text{Total field energy}) = \frac{d}{dR}\left(\frac{Q^2}{8\pi\epsilon_0 R}\right) = -\frac{Q^2}{8\pi\epsilon_0 R^2} \Rightarrow$$

$$\Delta(\text{Total field energy}) \simeq \frac{Q^2}{8\pi\epsilon_0 R^2}\Delta R \checkmark$$

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2) a) $I(t) = C_1 \frac{dV_1(t)}{dt} \qquad t > 0^+$

$$V_0 = RI(t) + V_1(t) = RC_1\frac{dV_1(t)}{dt} + V_1(t) \Rightarrow V_1(t) = V_0 + [V_1(t=0^+) - V_0]e^{-t/RC_1}$$

Note that the coefficients in the above expression for $V_1(t)$ are chosen such that at $t=0^+$ the capacitor's voltage is $V_1(t=0^+)$, while at $t=\infty$ the capacitor's voltage is V_0, that is, the battery's voltage.

$$I(t) = C_1\frac{dV_1(t)}{dt} = \frac{1}{R}[V_0 - V_1(t=0^+)]e^{-t/RC_1} = \frac{1}{R}\left(V_0 - \frac{C_0}{C_1}V_0\right)e^{-t/RC_1} \Rightarrow$$

$$I(t) = \frac{C_1 - C_0}{RC_1}V_0 e^{-t/RC_1}$$

b) Energy delivered to the circuit by battery $= \int_{t=0}^{\infty} V_0 I(t)\,dt =$

$$\frac{C_1 - C_0}{RC_1}V_0^2\int_0^\infty e^{-t/RC_1}dt = (C_1 - C_0)V_0^2$$

c) Energy consumed in resistor $= \int_0^\infty RI^2(t)\,dt = \frac{(C_1-C_0)^2}{RC_1^2}V_0^2\int_0^\infty e^{-2t/RC_1}dt$

$$= \frac{(C_1-C_0)^2}{2C_1}V_0^2$$

d) At $t=0$, the $\vec{E}$-field acting on each plate of the capacitor is $\frac{1}{2}E_0$, and the charge on each plate is Q_0; therefore, the effective force on each plate is $\frac{1}{2}Q_0E_0$, trying to pull the plates together. This force does mechanical work on the outside world when the distance between the plates is reduced from d_0 to d_1. The amount of this mechanical work is given by: Mechanical work on the outside world $= \frac{1}{2} Q_0 E_0 (d_0 - d_1)$

$$= \frac{1}{2}(C_0 V_0)(V_0/d_0)(d_0 - d_1) = \frac{1}{2} C_0 V_0^2 \left(1 - \frac{d_1}{d_0}\right) = \frac{1}{2} C_0 V_0^2 \left(1 - \frac{C_0}{C_1}\right)$$

$$= \frac{1}{2}\left(\frac{C_0}{C_1}\right)(C_1 - C_0) V_0^2$$

e) Energy delivered to capacitor by battery = energy delivered to the entire circuit − Energy consumed by the resistor =

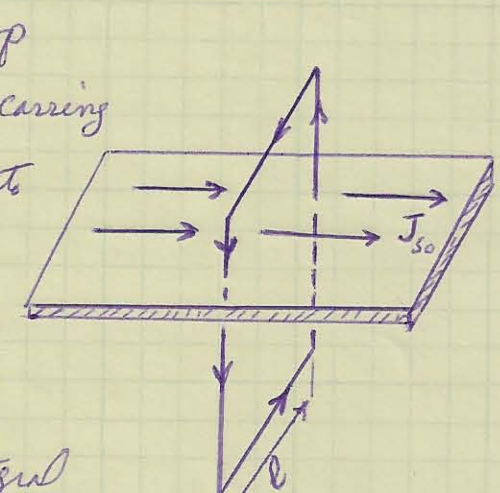
$$(C_1 - C_0) V_0^2 - \frac{(C_1 - C_0)^2}{2 C_1} V_0^2 = \frac{1}{2}\left(1 + \frac{C_0}{C_1}\right)(C_1 - C_0) V_0^2$$

(The same result may be obtained by computing $\int_0^\infty V_1(t) I(t)\, dt$ directly.)

(Energy delivered to capacitor by battery) − (Mechanical work done by capacitor on the outside world) $= \frac{1}{2}\left(1 + \frac{C_0}{C_1}\right)(C_1 - C_0) V_0^2 - \frac{1}{2}\left(\frac{C_0}{C_1}\right)(C_1 - C_0) V_0^2 = \frac{1}{2}(C_1 - C_0) V_0^2$.

This is equal to the change in the stored energy within the capacitor, namely, $W_1 - W_0 = \frac{1}{2} C_1 V_0^2 - \frac{1}{2} C_0 V_0^2 = \frac{1}{2}(C_1 - C_0) V_0^2$. ✓

3) a) Consider a rectangular loop perpendicular to the current-carrying sheet and also perpendicular to the direction of current density $\vec{J}_s$. From symmetry, $\vec{H}$ must be parallel to the current-carrying sheet and perpendicular to $\vec{J}_s$. Using the integral



form of Ampère's law, $\oint \vec{H} \cdot d\vec{\ell} = I$ over the rectangular loop, we find the contributions of the vertical legs to be zero, while the horizontal legs at the top and bottom contribute equally to the loop integral; therefore,

$$H_x(x,y,z>0)\,\ell - H_x(x,y,z<0)\,\ell = J_{s0}\,\ell \Rightarrow \vec{H}(\vec{r}) = \pm\tfrac{1}{2}J_{s0}\,\hat{x}$$

$H_x(x,y,z>0)$: Above the sheet

$H_x(x,y,z<0)$: Below the sheet

+ sign when $\vec{r}$ is above the sheet
− sign when $\vec{r}$ is below the sheet

b) From symmetry $\vec{A}(\vec{r})$ cannot have any dependence on x or y. Moreover, it must be directed along the y-axis, because $\vec{J}$ everywhere is along $\hat{y}$. Thus $\vec{A}(\vec{r}) = A_y(z)\,\hat{y}$. Consequently:

$$\vec{\nabla}\times\vec{A} = \vec{B} = \mu_0\vec{H} \Rightarrow -\frac{\partial}{\partial z}A_y(z)\,\hat{x} = \mu_0 H_x(\vec{r})\,\hat{x} \Rightarrow A_y(z) = \begin{cases} -\frac{1}{2}\mu_0 J_{s0}\,z & z\geq 0 \\ +\frac{1}{2}\mu_0 J_{s0}\,z & z\leq 0 \end{cases}$$

In compact form: $\boxed{\vec{A}(\vec{r}) = -\frac{1}{2}\mu_0 |z|\, J_{s0}\,\hat{y}}$ ← Note: $\vec{A}$ and $\vec{J}$ are in opposite directions because the integration constant has been ignored.

c) The magnetic field is the sum of the fields produced by the two current-carrying sheets. Therefore, $\vec{H}(\vec{r}) = \begin{cases} -J_{s0}\,\hat{x} & \leftarrow \text{Between the sheets} \\ 0 & \leftarrow \text{outside} \end{cases}$

d) Change in the stored magnetic field energy $= \frac{1}{2}\mu_0|\vec{H}|^2 a(d_1-d_0) = \frac{1}{2}\mu_0 J_{s0}^2\, a(d_1-d_0)$.

a: sheet area

Work done by the upper plate on the outside world $= (\frac{1}{2}\mu_0 J_{s0})(J_{s0})\,a(d_1-d_0)$

$\frac{1}{2}\mu_0 J_{s0}$: $\vec{B}$-field of lower plate on the upper plate

J_{s0}: Current density of upper plate

The total energy provided by the batteries must be the <u>sum</u> of the above energies, namely, $\mu_0 J_{s0}^2\, a(d_1-d_0)$.

The $\vec{E}$-field acting on the electrons in the upper sheet, when the sheet moves up at a velocity $v(t)$, is $\vec{v}(t)\times\vec{B} = -\frac{1}{2}\mu_0 J_{s0}\, v(t)\,\hat{y}$. This field, when integrated along the y-axis, yields the required voltage $V_1(t)$ to maintain the current in the upper sheet. The energy supplied by the upper battery is thus $\int_0^T V_1(t)I(t)\,dt = \frac{1}{2}\mu_0 J_{s0}^2\, a\int_0^T v(t)\,dt = \frac{1}{2}\mu_0 J_{s0}^2\, a(d_1-d_0)$. Here T is the time it takes for the distance between the plates to go from d_0 to d_1. As for the lower plate, it does <u>not</u> move and the $\vec{B}$ field acting on it does <u>not</u> change either. However, $\vec{E} = -\partial\vec{A}/\partial t$, generated by the movement of the upper plate, causes an identical voltage, $V_2(t) = V_1(t)$, in the lower plate. The energy supplied by $V_2(t)$ is thus $\frac{1}{2}\mu_0 J_{s0}^2\, a(d_1-d_0)$.