1) a) $\vec{E}(x,y,z=0) = -\frac{2Q}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}\hat{z} = \frac{-2Qd}{4\pi\epsilon_0 r^3}\hat{z} = -\frac{2Qd\,\hat{z}}{4\pi\epsilon_0(x^2+y^2+d^2)^{3/2}}$

b) $\psi(x,y,z=0) = \frac{Q}{4\pi\epsilon_0 r} - \frac{Q}{4\pi\epsilon_0 r} = 0$

Alternatively, since the E-field is $\perp$ to xy-plane, the integral of $\vec{E}\cdot d\vec{\ell}$ from any point in the xy-plane to ∞, taken along a path in the xy-plane, will be zero. By definition, this integral is the potential at that point; therefore, $\psi(x,y,0)=0$.

c) This is the so-called "method of images". The $\vec{E}$-field in the upper half-space $z>0$ is the same for systems in Fig. 1(a) and 1(b). The surface-charge density is proportional to the $\perp$ component of the $\vec{E}$-field at the conductor's surface. Thus

$$\sigma(x,y) = \epsilon_0 E_\perp(x,y,z=0) = -\frac{Qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

Note: The integral of $\sigma(x,y)$ over the entire xy-plane is equal to $-Q$.

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2) a) $\vec{m} = I_0 a^2 \vec{N}$ ← (Current × loop area, in the direction of the surface normal; Right-hand rule applies)

b) Let s be the cross-sectional area of the wire, and denote by ρ_0 the density of conduction electrons within the wire. Also assume that these electrons move at a constant velocity $\vec{V}$ along the length of the wire. In a time interval Δt, the charges move a distance $V\Delta t$. The volume of the charge going through a given cross-section is $sV\Delta t$, and the amount of charge is $\Delta Q = \rho_0 sV\Delta t$. Therefore, $I_0 = \frac{\Delta Q}{\Delta t} = \rho_0 sV$.

According to Lorentz law, the $\vec{B}$-field contribution to the force on charge q moving at velocity $\vec{V}$ is $\vec{F} = q\vec{V}\times\vec{B}$. For each side of the loop, the total amount of conduction electron charge is $q = aS\rho_0$. Therefore, $\vec{F} = aS\rho_0\vec{V}\times\vec{B}$.

On the two sides of the loop that are parallel to $\hat{x}$, $\vec{V}$ and $\vec{B} = B_0\hat{z}$ are orthogonal; therefore, $\vec{F}_{1,3} = \pm aS\rho_0 V B_0\hat{y} = \pm aI_0B_0\hat{y}$.

On the other two sides of the loop, there is an angle $90° - \theta$ between $\vec{V}$ and $\vec{B}$. Therefore, $\vec{F}_{2,4} = \pm aI_0B_0\cos\theta\,\hat{x}$. Here we have labelled the sides as 1, 2, 3, 4.

c) The net force on the loop is zero, because forces on opposite sides cancel out. As for the torque, the two forces along the $\hat{x}$-axis go through the center of the loop and, therefore, have no torque. The two forces along the $\hat{y}$-axis, namely, $\vec{F}_1$ and $\vec{F}_3$ are antiparallel, and are separated from each other by a distance $a\sin\theta$ along the $\hat{z}$-axis. The torque is along the $\hat{x}$-axis, its magnitude given by the force $\vec{F}_1$ (or $\vec{F}_3$) multiplied by the vertical separation between these forces:

$$\vec{T} = (a\sin\theta)\hat{z}\times\vec{F}_3 = a^2I_0B_0\sin\theta\,\hat{x} = |\vec{m}|\,B_0\sin\theta\,\hat{x} = \vec{m}\times\vec{B}$$

d) $\vec{B}(x,y,z) = B_0(y)\hat{z} \cong \left\{B_0(0) + \frac{d}{dy}B_0(y)\Big|_{y=0}\,y\right\}\hat{z} = [B_0(0) + B_0'(0)y]\hat{z}$

The Lorentz forces on sides 2 and 4 will continue to be equal and opposite (along $\hat{x}$ and $-\hat{x}$), and will, therefore, cancel out. The forces on sides 1 and 3, however, will differ because on side 1 the $\vec{B}$-field magnitude is $B_0(0) + \frac{1}{2} a \cos\theta \, B_0'(0)$, while on side 3 the $\vec{B}$-field magnitude is $B_0(0) - \frac{1}{2} a \cos\theta \, B_0'(0)$. We thus have:

$$\vec{F} = \vec{F}_1 + \vec{F}_3 = aI_0 \left[B_0(0) + \tfrac{1}{2} a \cos\theta \, B_0'(0)\right]\hat{y} - aI_0\left[B_0(0) - \tfrac{1}{2} a \cos\theta \, B_0'(0)\right]\hat{y}$$

$$= a^2 I_0 \cos\theta \, B_0'(0)\,\hat{y} = |\vec{m}|\, B_0'(0) \cos\theta \,\hat{y} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

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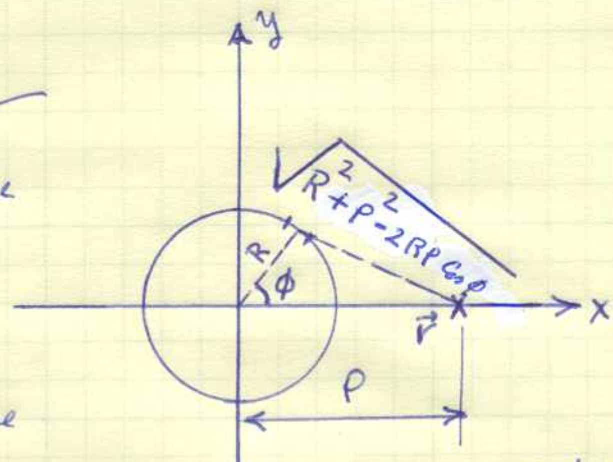
3) a) $\vec{J}_s = \left(\frac{I_0}{2\pi R}\right)\hat{z}$



b) For a single wire located at the azimuth ϕ, the distance to $\vec{r}$ is $\sqrt{R^2+\rho^2-2R\rho\cos\phi}$, as shown in the figure. The current in this wire is $(I_0/2\pi)\,d\phi$. Its vector potential at point $\vec{r}$ is known to be $\vec{A}(\vec{r}) = -\frac{\mu_0 (I_0/2\pi)\,d\phi}{2\pi} \ln\sqrt{R^2+\rho^2-2R\rho\cos\phi}\;\hat{z}$.

All we need to do then is to integrate over all the wires around the cylinder, that is,

$$\vec{A}(\vec{r}) = -\frac{\mu_0 I_0}{4\pi^2}\hat{z}\int_{\phi=0}^{2\pi} \ln\sqrt{R^2+\rho^2-2R\rho\cos\phi}\;d\phi = -\frac{\mu_0 I_0}{4\pi^2}\hat{z}\int_{\phi=0}^{\pi}\left[\ln R^2 + \ln\left(1+\frac{\rho^2}{R^2}-2\frac{\rho}{R}\cos\phi\right)\right]d\phi$$

$$= -\frac{\mu_0 I_0 \hat{z}}{4\pi^2}\left\{2\pi \ln R + \begin{cases} 0 & \rho \le R \\ 2\pi \ln(\rho/R) & \rho > R \end{cases}\right\} \Rightarrow \vec{A}(\vec{r}) = -\frac{\mu_0 I_0 \hat{z}}{2\pi}\begin{cases} \ln R; & \rho \le R \\ \ln\rho; & \rho > R \end{cases}$$

c) $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \mu_0 \vec{H}(\vec{r}) = -\frac{\partial A_3}{\partial \rho}\hat{\phi} \Rightarrow \vec{H}(\vec{r}) = \begin{cases} 0 & \rho \leq R \\ \frac{I_0}{2\pi\rho}\hat{\phi} & \rho > R \end{cases}$

The above $\vec{H}$ satisfies Ampère's law, namely, $\oint_{\text{Circle}} \vec{H}\cdot d\vec{\ell} = 2\pi\rho\left(\frac{I_0}{2\pi\rho}\right) = I_0$!

At the cylinder surface the $\vec{H}$-field just inside the surface is zero, and the field just outside is $(I/2\pi R)\hat{\phi}$. This discontinuity in the tangential H-field is exactly equal to the surface current density J_s, and perpendicular to it. Thus the boundary condition is satisfied.

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4) a) $\vec{B}(\vec{r},t) = \mu_0\left[\vec{H}_1(\vec{r},t) + H_2(\vec{r},t)\right] = \frac{\mu_0 I_0}{2\pi}\left(\frac{1}{\frac{d}{2}+x} + \frac{1}{\frac{d}{2}-x}\right)\sin(2\pi f t)\hat{z}$

Here x is the distance from the center-line. The expression can be slightly simplified to yield:

$$\vec{B}(\vec{r},t) = \frac{2\mu_0 I_0 d}{\pi(d^2-4x^2)}\sin(2\pi f t)\hat{z}; \qquad -\left(\tfrac{1}{2}d-R\right) \leq x \leq +\left(\tfrac{1}{2}d-R\right)$$

b) The two wires contribute equally to the flux, so we double the contribution of one wire to find the total flux:

$$\Phi(t) = 2\int_R^{d-R} \frac{\mu_0 I_0}{2\pi r}\sin(2\pi f t)\,dr = \frac{\mu_0 I_0}{\pi}\sin(2\pi f t)\ln r\Big|_R^{d-R} \Rightarrow$$

$$\Phi(t) = \frac{\mu_0 I_0}{\pi}\ln\left(\frac{d}{R}-1\right)\sin(2\pi f t)$$

c) $\Phi(t) = L\,I(t) \Rightarrow \boxed{L = \frac{\mu_0}{\pi}\ln\left(\frac{d}{R}-1\right)}$ Note: μ_0 and L have units of Henry/meter.

d) $\oint \vec{E}\cdot d\vec{\ell} = -\frac{\partial \Phi}{\partial t} \Rightarrow E(t) = -\frac{1}{2}\frac{\partial \Phi}{\partial t} = -\mu_0 I_0 f \ln\left(\frac{d}{R}-1\right)\cos(2\pi f t).$

The legs of the rectangle along the wire contribute equally; the other two legs cancel out.