Problem 1) Let the mirror acquire a velocity V along a direction that makes an angle θ with the x-axis within the xy-plane. Denoting by $\mathcal{E}'$ the energy of the light pulse after reflection, conservation of energy and momentum before and after reflection yields the following equations.

a) Relativistic treatment:

Energy conservation: $$\mathcal{E}+M_o c^2=\mathcal{E}'+M_o c^2/\sqrt{1-V^2/c^2} \qquad (1a)$$

Momentum conservation along x: $$\mathcal{E}/c=M_o V\cos\theta/\sqrt{1-V^2/c^2} \qquad (1b)$$

Momentum conservation along y: $$(\mathcal{E}'/c)+M_o V\sin\theta/\sqrt{1-V^2/c^2}=0 \qquad (1c)$$

These three equations must now be solved for the three unknowns, $\mathcal{E}'$, V, and θ. Dividing Eq.(1c) by Eq.(1b) yields: $\tan\theta=-\mathcal{E}'/\mathcal{E}$. Substituting $\mathcal{E}'=-\mathcal{E}\tan\theta$ in Eq.(1a) and solving for V, we find

$$\sqrt{1-V^2/c^2}=M_o c^2/[M_o c^2+(1+\tan\theta)\mathcal{E}], \qquad (2a)$$

$$V=c\sqrt{2M_o c^2(1+\tan\theta)\mathcal{E}+(1+\tan\theta)^2\mathcal{E}^2}/[M_o c^2+(1+\tan\theta)\mathcal{E}]. \qquad (2b)$$

The above expressions for V and $\sqrt{1-V^2/c^2}$ may now be placed into Eq.(1b) to yield

$$\mathcal{E}=\cos\theta\sqrt{2M_o c^2(1+\tan\theta)\mathcal{E}+(1+\tan\theta)^2\mathcal{E}^2} \quad\rightarrow\quad \tan\theta=-1/[1+(\mathcal{E}/M_o c^2)]. \qquad (3a)$$

Substitution into the preceding equations for $\mathcal{E}'$ and V then yields

$$\mathcal{E}'=\mathcal{E}/[1+(\mathcal{E}/M_o c^2)], \qquad (3b)$$

$$V=(\mathcal{E}/M_o c)\sqrt{2+2(\mathcal{E}/M_o c^2)+(\mathcal{E}/M_o c^2)^2}/[1+(\mathcal{E}/M_o c^2)+(\mathcal{E}/M_o c^2)^2]. \qquad (3c)$$

b) Non-relativistic treatment:

Energy conservation: $$\mathcal{E}=\mathcal{E}'+\tfrac{1}{2}M_o V^2 \qquad (4a)$$

Momentum conservation along x: $$\mathcal{E}/c=M_o V\cos\theta \qquad (4b)$$

Momentum conservation along y: $$(\mathcal{E}'/c)+M_o V\sin\theta=0 \qquad (4c)$$

These three equations must now be solved for the three unknowns, $\mathcal{E}'$, V, and θ. Dividing Eq.(4c) by Eq.(4b) yields: $\tan\theta=-\mathcal{E}'/\mathcal{E}$. Substituting for $\mathcal{E}$ and $\mathcal{E}'$ from Eqs.(4b) and (4c) into Eq.(4a), then solving for V, yields $V=2c(\cos\theta+\sin\theta)$. Placing this expression for V into Eq.(4b) and solving for $\tan\theta$, we find

$$\mathcal{E}/c=2M_o c(\cos\theta+\sin\theta)\cos\theta=2M_o c(1+\tan\theta)\cos^2\theta=2M_o c(1+\tan\theta)/(1+\tan^2\theta) \rightarrow$$

$$\tan\theta=(M_o c^2/\mathcal{E})[1-\sqrt{1+2(\mathcal{E}/M_o c^2)-(\mathcal{E}/M_o c^2)^2}], \qquad (5a)$$

$$\mathcal{E}'=-\mathcal{E}\tan\theta, \qquad (5b)$$

$$V=2c(1+\tan\theta)/\sqrt{1+\tan^2\theta}. \qquad (5c)$$

Problem 2) a) Snell's law: $k_x^{(i)} = k_x^{(t)}$. Below, both $k_x^{(i)}$ and $k_x^{(t)}$ will be written as k_x.

Dispersion relation in free space: $\boldsymbol{k}^{(i)2} = k_x^{(i)2} + k_z^{(i)2} = (\omega/c)^2$; therefore, $k_z^{(i)} = \pm\sqrt{(\omega/c)^2 - k_x^2}$. Note that, in general, the square root will yield a complex number. Either the plus sign or the minus sign (but not both) should be used for the square root.

Dispersion relation in material medium: $\boldsymbol{k}^{(t)2} = k_x^{(t)2} + k_z^{(t)2} = (\omega/c)^2 \mu(\omega)\varepsilon(\omega)$. Since $k_x^{(i)} = k_x^{(t)} = k_x$ and $\mu(\omega) = 1$, we will have $k_z^{(t)} = \pm\sqrt{(\omega/c)^2 \varepsilon(\omega) - k_x^2}$. As before, the square root will, in general, yield a complex number. Either the plus sign or the minus sign (but not both) should be used.

b) Maxwell's first equation: $\boldsymbol{k}^{(i)} \cdot \boldsymbol{E}_o^{(i)} = 0 \rightarrow k_x^{(i)} E_{xo}^{(i)} + k_z^{(i)} E_{zo}^{(i)} = 0 \rightarrow E_{zo}^{(i)} = -k_x E_{xo}^{(i)} / k_z^{(i)}$.

transmitted beam: $\boldsymbol{k}^{(t)} \cdot \boldsymbol{E}_o^{(t)} = 0 \rightarrow E_{zo}^{(t)} = -k_x E_{xo}^{(t)} / k_z^{(t)}$.

Maxwell's third equation; incident beam: $\boldsymbol{k}^{(i)} \times \boldsymbol{E}_o^{(i)} = \mu_o \omega \boldsymbol{H}_o^{(i)} \rightarrow H_{xo}^{(i)} = -k_z^{(i)} E_{yo}^{(i)} / (\mu_o \omega)$;

$$H_{yo}^{(i)} = [k_z^{(i)} E_{xo}^{(i)} - k_x E_{zo}^{(i)}] / (\mu_o \omega) = \varepsilon_o \omega E_{xo}^{(i)} / k_z^{(i)}; \qquad H_{zo}^{(i)} = k_x E_{yo}^{(i)} / (\mu_o \omega).$$

transmitted beam: $\boldsymbol{k}^{(t)} \times \boldsymbol{E}_o^{(t)} = \mu_o \mu(\omega) \omega \boldsymbol{H}_o^{(t)} \rightarrow H_{xo}^{(t)} = -k_z^{(t)} E_{yo}^{(t)} / (\mu_o \omega)$;

$$H_{yo}^{(t)} = [k_z^{(t)} E_{xo}^{(t)} - k_x E_{zo}^{(t)}] / (\mu_o \omega) = \varepsilon_o \varepsilon \omega E_{xo}^{(t)} / k_z^{(t)}; \qquad H_{zo}^{(t)} = k_x E_{yo}^{(t)} / (\mu_o \omega).$$

c) Continuity equations for the tangential E- and H-fields at the $z = 0$ interface:

$$p\text{-polarization:} \quad \begin{cases} E_{xo}^{(i)} = E_{xo}^{(t)} \\ H_{yo}^{(i)} = H_{yo}^{(t)} \rightarrow \ \varepsilon_o \omega E_{xo}^{(i)} / k_z^{(i)} = \varepsilon_o \varepsilon \omega E_{xo}^{(t)} / k_z^{(t)} \ \rightarrow \ k_z^{(t)} = \varepsilon(\omega) k_z^{(i)}. \end{cases}$$

$$s\text{-polarization:} \quad \begin{cases} E_{yo}^{(i)} = E_{yo}^{(t)} \\ H_{xo}^{(i)} = H_{xo}^{(t)} \ \rightarrow \ k_z^{(t)} = k_z^{(i)}. \end{cases}$$

d) For the case of p-polarization, satisfying the boundary conditions *without* a reflected wave requires that $k_z^{(t)} = \varepsilon(\omega) k_z^{(i)}$. Substituting in this equation the expressions for $k_z^{(i)}$ and $k_z^{(t)}$ obtained in part (a), we find

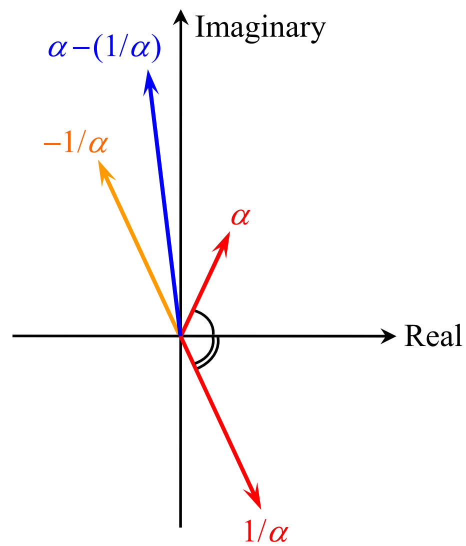
$$(\omega/c)^2 \varepsilon(\omega) - k_x^2 = \varepsilon^2(\omega)[(\omega/c)^2 - k_x^2] \ \rightarrow \ k_x = \pm(\omega/c)\sqrt{\varepsilon(\omega)/[1+\varepsilon(\omega)]}.$$

For the case of s-polarization, the boundary conditions in the absence of a reflected wave will be satisfied only when $k_z^{(i)} = k_z^{(t)}$, which is *impossible* so long as $\varepsilon(\omega) \neq 1$.

e) **Case i**: $\varepsilon' > 0$, $\varepsilon'' = 0$. Here $\varepsilon' = n^2$, where n is the real-valued, positive refractive index of the material medium. When the reflection coefficient for p-polarized light vanishes, we will have $k_x = \pm(\omega/c)\sqrt{n^2/(1+n^2)} = \pm(\omega/c)\sin\theta_B$ where $\theta_B = \tan^{-1} n$ is the Brewster angle. Substituting for k_x in the expressions for $k_z^{(i)}$ and $k_z^{(t)}$, we find $k_z^{(i)} = -(\omega/c)\cos\theta_B$ and $k_z^{(t)} = -(n^2\omega/c)\cos\theta_B$. Both the incident and transmitted plane-waves are thus homogeneous; they propagate downward, along the negative z-axis, and satisfy the condition $k_z^{(t)} = \varepsilon(\omega) k_z^{(i)}$ obtained in part (c) for p-polarized light.

Case ii: $\varepsilon' < -1$, $\varepsilon'' = 0$. When the reflection coefficient for p-polarized light vanishes, we will have $k_x = \pm(\omega/c)\sqrt{|\varepsilon'|/(|\varepsilon'|-1)}$, which is a real-valued number with a magnitude greater than ω/c. Substitution for k_x in the expressions for $k_z^{(\mathrm{i})}$ and $k_z^{(\mathrm{t})}$ yields $k_z^{(\mathrm{i})} = \mathrm{i}(\omega/c)/\sqrt{|\varepsilon'|-1}$ and $k_z^{(\mathrm{t})} = -\mathrm{i}(\omega/c)|\varepsilon'|/\sqrt{|\varepsilon'|-1}$. Both the incident and transmitted waves are thus *evanescent*, with real-valued k_x and imaginary k_z; they attenuate away from the interface along the $\pm z$-axis, and satisfy the required condition $k_z^{(\mathrm{t})} = \varepsilon(\omega) k_z^{(\mathrm{i})}$ obtained for *p*-polarized light in part (c). The time-averaged Poynting vector $\langle \boldsymbol{S} \rangle = ½\mathrm{Real}(\boldsymbol{E} \times \boldsymbol{H}^*)$ can be readily calculated from the (E_x, E_z, H_y) fields given in part (b). The energy is seen to flow along k_x in the free space, and along $-k_x$ inside the medium. On both sides of the interface, the time-averaged energy flux along the z-axis is zero. This excited surface-wave, residing partly in the free space and partly in the material medium, is known as a *surface plasmon polariton*.

Case iii: $\varepsilon' < 0$, $\varepsilon'' > 0$. In this case $k_x = \pm(\omega/c)\sqrt{(\varepsilon' + \mathrm{i}\varepsilon'')/(1 + \varepsilon' + \mathrm{i}\varepsilon'')}$ is complex-valued. Substitution for k_x in the expressions for $k_z^{(\mathrm{i})}$ and $k_z^{(\mathrm{t})}$ yields $k_z^{(\mathrm{i})} = (\omega/c)(1 + \varepsilon' + \mathrm{i}\varepsilon'')^{-½}$ and $k_z^{(\mathrm{t})} = (\omega/c)(\varepsilon' + \mathrm{i}\varepsilon'')(1 + \varepsilon' + \mathrm{i}\varepsilon'')^{-½}$. The complex square root $(1 + \varepsilon' + \mathrm{i}\varepsilon'')^{-½}$ is chosen to give $k_z^{(\mathrm{i})}$ a *positive* imaginary part. Note that our choice of signs for $k_z^{(\mathrm{i})}$ and $k_z^{(\mathrm{t})}$ satisfies the required condition $k_z^{(\mathrm{t})} = \varepsilon(\omega) k_z^{(\mathrm{i})}$ obtained in part (c). We must prove that the imaginary parts of $k_z^{(\mathrm{i})}$ and $k_z^{(\mathrm{t})}$ always have opposite signs. To this end, note that $(1+\varepsilon)^{-½} + \varepsilon(1+\varepsilon)^{-½} = (1+\varepsilon)^{½}$; therefore, $\varepsilon(1+\varepsilon)^{-½} = (1+\varepsilon)^{½} - (1+\varepsilon)^{-½}$. From the complex-plane diagram below it must be clear that, for *any* complex number α, the imaginary parts of $\alpha - (1/\alpha)$ and $(1/\alpha)$ always have opposite signs, which completes the proof. The *evanescent* plane-wave in the free space region decays exponentially along the imaginary part of $k_x^{(\mathrm{i})}\hat{\boldsymbol{x}} + k_z^{(\mathrm{i})}\hat{\boldsymbol{z}}$, which points away from the interface. The *inhomogeneous* plane-wave in the material medium also decays exponentially away from the interface, this one along the imaginary part of $k_x^{(\mathrm{t})}\hat{\boldsymbol{x}} + k_z^{(\mathrm{t})}\hat{\boldsymbol{z}}$.



Typical metals at optical frequencies have large negative values of ε' in addition to small positive values of ε''. For these, the *surface plasmon polariton* wave will have a k_x value slightly greater than unity (in magnitude), with a small imaginary component. The evanescent wave in the free space decays rather slowly along the z-axis, whereas the inhomogeneous wave in the metal decays quite rapidly away from the interface. The plasmonic wave is thus confined to a thin layer at the surface of the metallic medium. The time-averaged Poynting vector $\langle \boldsymbol{S} \rangle = ½\mathrm{Real}(\boldsymbol{E} \times \boldsymbol{H}^*)$ can be readily calculated from the (E_x, E_z, H_y) fields given in part (b). The horizontal energy flux, $\langle S_x \rangle$, is seen to be along $\mathrm{Real}(k_x)$ in the free space, and along $\mathrm{Real}(-k_x)$ inside the medium. On both sides of the interface, vertical energy flux, $\langle S_z \rangle$, is downward, i.e., points along the negative z-axis. Such plasmonic waves are generally long-range, because ε'' is fairly small and the losses are confined to an exceedingly thin layer at the surface of the metallic medium.

Case iv: $\varepsilon' > 0$, $\varepsilon'' > 0$. This case is similar to case (iii), with the following exceptions: The magnitude of k_x is generally *less* than unity, with an imaginary part that may be large or small, depending on the relative values of ε' and ε''. For a low-loss medium, where ε'' is fairly small, the exponential decay of the wave inside the medium (away from the interface) is rather slow, resulting in a large penetration depth. The horizontal energy flux, $<S_x>$, is in the direction of Real(k_x), both in the free space region and inside the material medium. The vertical energy flux, $<S_z>$, always pointing along the negative z-axis, is large, irrespective of whether ε'' is large or small. The wave is thus very different from a *surface plasmon polariton*, despite similarities in their mathematical structure. When integrated over the penetration depth, the lost energy will be substantial, even for small values of ε''. Therefore, a p-polarized wave-packet comprising an evanescent plane-wave in the free space region and an inhomogeneous plane-wave in a medium having $\varepsilon' > 0$, $\varepsilon'' > 0$, cannot behave similarly to a long-range surface wave; too much energy is dissipated within its penetration depth, and not enough energy is transported parallel to the surface of the medium.

Problem 3) a) Using the dispersion relation, $\boldsymbol{k}^2 = k_x^2 + k_z^2 = (\omega/c)^2 \mu(\omega)\varepsilon(\omega)$, and the fact that the x-component of $\boldsymbol{k}$ is given by $k_x = (\omega/c)n(\omega)\sin\theta$, we write

$$k_z = \sqrt{(\omega/c)^2 n(\omega)^2 - k_x^2} = (\omega/c)n(\omega)\cos\theta. \tag{1}$$

Maxwell's 1st equation: $\boldsymbol{k}_1\cdot\boldsymbol{E}_1 = 0 \rightarrow k_x E_{x1} + k_z E_{z1} = 0 \rightarrow E_{z1} = -k_x E_{x1}/k_z \rightarrow E_{z1} = -(\tan\theta)E_{x1}$.

$$\text{Similarly, } \boldsymbol{k}_2\cdot\boldsymbol{E}_2 = 0 \rightarrow E_{z2} = (\tan\theta)E_{x2}. \tag{2}$$

Maxwell's 3rd equation: $\boldsymbol{k}_1\times\boldsymbol{E}_1 = \mu_o\mu(\omega)\omega\boldsymbol{H}_1 \rightarrow H_{x1} = -k_z E_{y1}/(\mu_o\omega) \rightarrow H_{x1} = -n(\omega)E_{y1}\cos\theta/Z_o$;

$$H_{y1} = (k_z E_{x1} - k_x E_{z1})/(\mu_o\omega) = n(\omega)E_{x1}/(Z_o\cos\theta); \quad H_{z1} = k_x E_{y1}/(\mu_o\omega) = n(\omega)E_{y1}\sin\theta/Z_o. \tag{3a}$$

$$\text{Similarly, } H_{x2} = n(\omega)E_{y2}\cos\theta/Z_o; \quad H_{y2} = -n(\omega)E_{x2}/(Z_o\cos\theta); \quad H_{z2} = n(\omega)E_{y2}\sin\theta/Z_o. \tag{3b}$$

b) Setting $E_{x2} = E_{x1}$ and $E_{y2} = E_{y1}$ for an even mode, the superposition of the two plane-waves produces the following fields throughout the waveguide:

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{r},t) &= \text{Real}\{\boldsymbol{E}_1\exp[i(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega t)] + \boldsymbol{E}_2\exp[i(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega t)]\}\\
&= \text{Real}\{[\boldsymbol{E}_1\exp(ik_z z) + \boldsymbol{E}_2\exp(-ik_z z)]\exp[i(k_x x-\omega t)]\}\\
&= \text{Real}\{\{E_{x1}[\exp(ik_z z)+\exp(-ik_z z)]\hat{\boldsymbol{x}} + E_{y1}[\exp(ik_z z)+\exp(-ik_z z)]\hat{\boldsymbol{y}}\\
&\qquad -\tan\theta\, E_{x1}[\exp(ik_z z)-\exp(-ik_z z)]\hat{\boldsymbol{z}}\}\exp[i(k_x x-\omega t)]\}\\
&= 2E_{x1}\cos(k_z z)\cos(k_x x-\omega t)\hat{\boldsymbol{x}} + 2E_{y1}\cos(k_z z)\cos(k_x x-\omega t)\hat{\boldsymbol{y}} + 2\tan\theta\, E_{x1}\sin(k_z z)\sin(k_x x-\omega t)\hat{\boldsymbol{z}}.
\end{aligned} \tag{4a}$$

$$\begin{aligned}
\boldsymbol{H}(\boldsymbol{r},t) &= \text{Real}\{\boldsymbol{H}_1\exp[i(\boldsymbol{k}_1\cdot\boldsymbol{r}-\omega t)] + \boldsymbol{H}_2\exp[i(\boldsymbol{k}_2\cdot\boldsymbol{r}-\omega t)]\}\\
&= 2Z_o^{-1}n(\omega)[E_{y1}\cos\theta\sin(k_z z)\sin(k_x x-\omega t)\hat{\boldsymbol{x}} - (E_{x1}/\cos\theta)\sin(k_z z)\sin(k_x x-\omega t)\hat{\boldsymbol{y}}\\
&\qquad + E_{y1}\sin\theta\cos(k_z z)\cos(k_x x-\omega t)\hat{\boldsymbol{z}}].
\end{aligned} \tag{4b}$$

c) At the surface of the conductor, there cannot be any tangential E- or perpendicular B-fields, which means that $E_x = E_y = H_z = 0$ at $z = \pm d/2$. This is possible only when $\cos(\pm\tfrac{1}{2}k_z d) = 0$, that is,

$$\tfrac{1}{2}(\omega d/c)n(\omega)\cos\theta = (m+\tfrac{1}{2})\pi \quad\rightarrow\quad \cos\theta_m = (m+\tfrac{1}{2})\lambda_o/[n(\omega)d], \tag{5}$$

where the vacuum wavelength, $\lambda_o = 2\pi c/\omega$, has been used. The mode can exist when $\cos\theta < 1$. The smallest possible value of the integer m being zero, it is necessary to have $d > \frac{1}{2}\lambda_o/n(\omega)$ to ensure the existence of at least one even mode. The even mode is said to be "cut-off" when the slab thickness d happens to be below $\frac{1}{2}\lambda_o/n(\omega)$. For single mode operation, d must be in the following range:

$$\tfrac{1}{2}\lambda_o/n(\omega) < d < \tfrac{3}{2}\lambda_o/n(\omega). \qquad (6)$$

With regard to the polarization state of the guided mode, two possibilities exist:

i) *p*-polarized mode (also called transverse magnetic, TM, mode): $E_{x1} \neq 0$, $E_{y1} = 0$.

ii) *s*-polarized mode (also called transverse electric, TE, mode): $E_{x1} = 0$, $E_{y1} \neq 0$.

In the case of *even* modes currently under consideration, the preceding statements with regard to cut-off and single-mode operation apply to *both* TE and TM modes.

d) According to Maxwell's 1st equation, $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho_{\text{free}}$, the surface charge density is equal to the perpendicular D-field, $\varepsilon_o \varepsilon E_\perp$, at the surface of a perfect conductor. We thus have:

m^{th} *p*-polarized even mode:

$$\sigma_s(x, z = d/2, t) = -\varepsilon_o \varepsilon E_z(x, z = d/2, t) = 2(-1)^{m+1}\varepsilon_o n^2(\omega)\tan\theta_m E_{x1}\sin(k_x^{(m)}x - \omega t). \qquad (7a)$$

m^{th} *s*-polarized even mode:

$$\sigma_s(x, z = d/2, t) = 0. \qquad (7b)$$

Also, according to Maxwell's 2nd equation, $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{J}_{\text{free}} + \partial\boldsymbol{D}/\partial t$, the surface current density of a perfect conductor is equal but perpendicular to the tangential magnetic field, $\boldsymbol{H}_{||}$. Therefore,

m^{th} *p*-polarized even mode:

$$\boldsymbol{J}_s(x, z = d/2, t) = H_y(x, z = d/2, t)\hat{\boldsymbol{x}} = 2(-1)^{m+1}n(\omega)(E_{x1}/Z_o\cos\theta_m)\sin(k_x^{(m)}x - \omega t)\hat{\boldsymbol{x}}. \qquad (8a)$$

m^{th} *s*-polarized even mode:

$$\boldsymbol{J}_s(x, z = d/2, t) = -H_x(x, z = d/2, t)\hat{\boldsymbol{y}} = 2(-1)^{m+1}n(\omega)(E_{y1}/Z_o)\cos\theta_m\sin(k_x^{(m)}x - \omega t)\hat{\boldsymbol{y}}. \qquad (8b)$$

It may be readily verified that the above distributions satisfy the charge-current continuity equation, $\boldsymbol{\nabla}\cdot\boldsymbol{J}_s + \partial\sigma_s/\partial t = 0$.

e) Setting $E_{x2} = -E_{x1}$ and $E_{y2} = -E_{y1}$ for an odd mode, the superposition of the two plane-waves produces the following fields throughout the waveguide:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= \text{Real}\{\boldsymbol{E}_1\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega t)] + \boldsymbol{E}_2\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega t)]\}\\ &= \text{Real}\{\{E_{x1}[\exp(\mathrm{i}k_z z) - \exp(-\mathrm{i}k_z z)]\hat{\boldsymbol{x}} + E_{y1}[\exp(\mathrm{i}k_z z) - \exp(-\mathrm{i}k_z z)]\hat{\boldsymbol{y}}\\ &\qquad - \tan\theta E_{x1}[\exp(\mathrm{i}k_z z) + \exp(-\mathrm{i}k_z z)]\hat{\boldsymbol{z}}\}\exp[\mathrm{i}(k_x x - \omega t)]\}\\ &= -2[E_{x1}\sin(k_z z)\sin(k_x x - \omega t)\hat{\boldsymbol{x}} + E_{y1}\sin(k_z z)\sin(k_x x - \omega t)\hat{\boldsymbol{y}} + \tan\theta E_{x1}\cos(k_z z)\cos(k_x x - \omega t)\hat{\boldsymbol{z}}].\end{aligned} \qquad (9a)$$

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{r},t) &= \text{Real}\{\boldsymbol{H}_1\exp[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{r} - \omega t)] + \boldsymbol{H}_2\exp[\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{r} - \omega t)]\}\\ &= -2Z_o^{-1}n(\omega)[E_{y1}\cos\theta\cos(k_z z)\cos(k_x x - \omega t)\hat{\boldsymbol{x}} - (E_{x1}/\cos\theta)\cos(k_z z)\cos(k_x x - \omega t)\hat{\boldsymbol{y}}\\ &\qquad + E_{y1}\sin\theta\sin(k_z z)\sin(k_x x - \omega t)\hat{\boldsymbol{z}}].\end{aligned} \qquad (9b)$$

At the conductors' surfaces, $z=\pm d/2$, where $E_x=E_y=H_z=0$, we must have $\sin(\pm\frac{1}{2}k_z d)=0$, i.e.,

$$\tfrac{1}{2}(\omega d/c)n(\omega)\cos\theta = m\pi \quad \rightarrow \quad \cos\theta_m = m\lambda_o/[n(\omega)d]. \tag{10}$$

In this case the lowest-order mode, corresponding to $m=0$, obtains when $\theta_m=90°$. However, we now have $k_x=(\omega/c)n(\omega)$ and $k_z=0$. Under these circumstances, in accordance with Eqs.(9a) and (9b), E_x, E_y, H_x, and H_z will identically vanish throughout the slab. The only surviving fields are E_z and H_y, which go to infinity unless one recognizes that, by allowing E_x to approach zero when $\theta\rightarrow 90°$, E_z and H_y could attain finite values, namely,

$$\boldsymbol{E}(\boldsymbol{r},t) = E_z\cos(k_x^{(0)}x-\omega t)\hat{\boldsymbol{z}}; \qquad (m=0), \tag{11a}$$

$$\boldsymbol{H}(\boldsymbol{r},t) = -n(\omega)(E_z/Z_o)\cos(k_x^{(0)}x-\omega t)\hat{\boldsymbol{y}}; \qquad (m=0). \tag{11b}$$

This *p*-polarized (TM) mode always exists, no matter how thin the slab may be. Taking note of the fact that $\cos\theta_m \leq 1$ for any value of m, the condition for *p*-polarized *single-mode* operation in the $m=0$ guided mode is $d<\lambda_o/n(\omega)$.

For odd modes that are *s*-polarized (TE), the first possibility for propagation is $m=1$, in which case single-mode operation occurs when $\lambda_o/n(\omega)<d<2\lambda_o/n(\omega)$. The cut-off for *odd* TE modes occurs below $d=\lambda_o/n(\omega)$.

At the surface of the upper conductor which is in contact with the dielectric slab, surface charge and current densities for odd modes are found to be:

m^{th} odd *p*-polarized mode ($m\neq 0$):

$$\sigma_s(x,z=d/2,t) = -\varepsilon_o\varepsilon E_z(x,z=d/2,t) = 2(-1)^m\varepsilon_o n^2(\omega)\tan\theta_m E_{x1}\cos(k_x^{(m)}x-\omega t), \tag{12a}$$

$$\boldsymbol{J}_s(x,z=d/2,t) = H_y(x,z=d/2,t)\hat{\boldsymbol{x}} = 2(-1)^m n(\omega)(E_{x1}/Z_o\cos\theta_m)\cos(k_x^{(m)}x-\omega t)\hat{\boldsymbol{x}}. \tag{12b}$$

m^{th} odd *s*-polarized mode ($m\neq 0$):

$$\sigma_s(x,z=d/2,t)=0, \tag{13a}$$

$$\boldsymbol{J}_s(x,z=d/2,t) = -H_x(x,z=d/2,t)\hat{\boldsymbol{y}} = 2(-1)^m n(\omega)(E_{y1}/Z_o)\cos\theta_m\cos(k_x^{(m)}x-\omega t)\hat{\boldsymbol{y}}. \tag{13b}$$

Once again, it is easy to verify the satisfaction of the charge-current continuity equation for the above distributions.
