Problem 1)

a) Incident beam:

$$\boldsymbol{k}^{\mathrm{i}} = (\omega/c)(\sin\theta_{\mathrm{B}}\hat{\boldsymbol{x}} - \cos\theta_{\mathrm{B}}\hat{\boldsymbol{z}})\,, \tag{1a}$$

$$\boldsymbol{E}_{\mathrm{o}}^{\mathrm{i}} = E_{\mathrm{p}}^{\mathrm{i}}(\cos\theta_{\mathrm{B}}\hat{\boldsymbol{x}} + \sin\theta_{\mathrm{B}}\hat{\boldsymbol{z}}), \tag{1b}$$

$$\boldsymbol{H}_{\mathrm{o}}^{\mathrm{i}} = \boldsymbol{k}^{\mathrm{i}} \times \boldsymbol{E}_{\mathrm{o}}^{\mathrm{i}}/(\mu_{\mathrm{o}}\omega) = -(E_{\mathrm{p}}^{\mathrm{i}}/Z_{\mathrm{o}})\,\hat{\boldsymbol{y}}, \tag{1c}$$

$$\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r},t) = E_{\mathrm{p}}^{\mathrm{i}}(\cos\theta_{\mathrm{B}}\hat{\boldsymbol{x}} + \sin\theta_{\mathrm{B}}\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - z\cos\theta_{\mathrm{B}} - ct)], \tag{1d}$$

$$\boldsymbol{H}^{\mathrm{i}}(\boldsymbol{r},t) = -(E_{\mathrm{p}}^{\mathrm{i}}/Z_{\mathrm{o}})\,\hat{\boldsymbol{y}}\,\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - z\cos\theta_{\mathrm{B}} - ct)]. \tag{1e}$$

Transmitted beam:

$$\boldsymbol{k}^{\mathrm{t}} = (n\,\omega/c)(\sin\theta'_{\mathrm{B}}\,\hat{\boldsymbol{x}} - \cos\theta'_{\mathrm{B}}\,\hat{\boldsymbol{z}})\,, \tag{2a}$$

$$\boldsymbol{E}_{\mathrm{o}}^{\mathrm{t}} = E_{\mathrm{p}}^{\mathrm{t}}(\cos\theta'_{\mathrm{B}}\,\hat{\boldsymbol{x}} + \sin\theta'_{\mathrm{B}}\,\hat{\boldsymbol{z}}), \tag{2b}$$

$$\boldsymbol{H}_{\mathrm{o}}^{\mathrm{t}} = \boldsymbol{k}^{\mathrm{t}} \times \boldsymbol{E}_{\mathrm{o}}^{\mathrm{t}}/(\mu_{\mathrm{o}}\mu\,\omega) = -(n\,E_{\mathrm{p}}^{\mathrm{t}}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}, \tag{2c}$$

$$\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t) = E_{\mathrm{p}}^{\mathrm{t}}(\cos\theta'_{\mathrm{B}}\,\hat{\boldsymbol{x}} + \sin\theta'_{\mathrm{B}}\,\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(xn\,\sin\theta'_{\mathrm{B}} - zn\cos\theta'_{\mathrm{B}} - ct)]\,, \tag{2d}$$

$$\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r},t) = -(nE_{\mathrm{p}}^{\mathrm{t}}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}\,\exp[\mathrm{i}(\omega/c)(xn\,\sin\theta'_{\mathrm{B}} - zn\cos\theta'_{\mathrm{B}} - ct)]. \tag{2e}$$

b) At the $z=0$ interface we must have $\sin\theta_{\mathrm{B}} = n\sin\theta'_{\mathrm{B}}$ (Snell's law), so that the exponential factors will match. Also, continuity of the tangential *E*-field, E_x, yields $E_{\mathrm{p}}^{\mathrm{i}}\cos\theta_{\mathrm{B}} = E_{\mathrm{p}}^{\mathrm{t}}\cos\theta'_{\mathrm{B}}$, while the continuity of the tangential *H*-field, H_y, yields $E_{\mathrm{p}}^{\mathrm{i}} = n\,E_{\mathrm{p}}^{\mathrm{t}}$. Combining the last two equations, we find $n\cos\theta_{\mathrm{B}} = \cos\theta'_{\mathrm{B}}$. This equation together with Snell's law may now be solved for the two unknowns, θ_{B} and θ'_{B}, yielding $\tan\theta_{\mathrm{B}} = n$ and $\tan\theta'_{\mathrm{B}} = 1/n$. The transmitted *E*- and *H*-fields may now be written as follows:

$$\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t) = E_{\mathrm{p}}^{\mathrm{i}}(\cos\theta_{\mathrm{B}}\,\hat{\boldsymbol{x}} + n^{-2}\sin\theta_{\mathrm{B}}\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - zn^2\cos\theta_{\mathrm{B}} - ct)], \tag{3a}$$

$$\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r},t) = -(E_{\mathrm{p}}^{\mathrm{i}}/Z_{\mathrm{o}})\hat{\boldsymbol{y}}\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - zn^2\cos\theta_{\mathrm{B}} - ct)]. \tag{3b}$$

c) In the incidence medium, $\boldsymbol{D}^{\mathrm{i}}(\boldsymbol{r},t) = \varepsilon_{\mathrm{o}}\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r},t)$. Therefore, at $z=0^+$ we have

$$\boldsymbol{D}^{\mathrm{i}}(x,y,z=0^+,t) = \varepsilon_{\mathrm{o}}E_{\mathrm{p}}^{\mathrm{i}}(\cos\theta_{\mathrm{B}}\hat{\boldsymbol{x}} + \sin\theta_{\mathrm{B}}\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - ct)]. \tag{4}$$

In the dielectric medium, however, $\boldsymbol{D}^{\mathrm{t}}(\boldsymbol{r},t) = \varepsilon_{\mathrm{o}}\varepsilon\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t) = \varepsilon_{\mathrm{o}}n^2\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t)$. Thus at $z=0^-$ we have

$$\boldsymbol{D}^{\mathrm{t}}(x,y,z=0^-,t) = \varepsilon_{\mathrm{o}}n^2E_{\mathrm{p}}^{\mathrm{i}}(\cos\theta_{\mathrm{B}}\,\hat{\boldsymbol{x}} + n^{-2}\sin\theta_{\mathrm{B}}\hat{\boldsymbol{z}})\exp[\mathrm{i}(\omega/c)(x\sin\theta_{\mathrm{B}} - ct)]. \tag{5}$$

Clearly then $D_z^{\mathrm{t}}(x,y,z=0^-,t) = D_z^{\mathrm{i}}(x,y,z=0^+,t)$.

d) Incident beam: $<\boldsymbol{S}^{\text{i}}(\boldsymbol{r},t)> = \frac{1}{2}\text{Re}[\boldsymbol{E}^{\text{i}}(\boldsymbol{r},t)\times\boldsymbol{H}^{\text{i}*}(\boldsymbol{r},t)] = \dfrac{|E_{\text{p}}^{\text{i}}|^2}{2Z_{\text{o}}}(\sin\theta_{\text{B}}\hat{\boldsymbol{x}} - \cos\theta_{\text{B}}\hat{\boldsymbol{z}}).$ (6)

Transmitted beam: $<\boldsymbol{S}^{\text{t}}(\boldsymbol{r},t)> = \frac{1}{2}\text{Re}[\boldsymbol{E}^{\text{t}}(\boldsymbol{r},t)\times\boldsymbol{H}^{\text{t}*}(\boldsymbol{r},t)] = \dfrac{n|E_{\text{p}}^{\text{t}}|^2}{2Z_{\text{o}}}(\sin\theta'_{\text{B}}\,\hat{\boldsymbol{x}} - \cos\theta'_{\text{B}}\,\hat{\boldsymbol{z}}).$ (7)

Note that both Poynting vectors are aligned with their corresponding k-vector. However, since $E_{\text{p}}^{\text{i}} = nE_{\text{p}}^{\text{t}}$, the time-averaged Poynting vector of the incident beam is n times greater than that of the transmitted beam. Nevertheless, the cross-sectional areas of the two beams are in the ratio of $\cos\theta'_{\text{B}}/\cos\theta_{\text{B}}$, which is also equal to n. Therefore, the rate-of-flow of energy per unit time along the propagation direction is the same for the incident and transmitted beams, as required by energy conservation.

e) In the absence of free charge, Maxwell's 1st equation is $\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t) = 0$. Since $\boldsymbol{D} = \varepsilon_{\text{o}}\boldsymbol{E} + \boldsymbol{P}$ and, by definition, $\rho_{\text{bound}}^{(\text{e})}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t)$, we have $\varepsilon_{\text{o}}\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t) = \rho_{\text{bound}}^{(\text{e})}(\boldsymbol{r},t)$. Thus the discontinuity of $\varepsilon_0 E_z$ at the $z=0$ interface is equal to the bound surface-charge-density. Using Eqs.(1d) and (3a) we find

$$\sigma_s^{(\text{bound})}(x,y,z=0,t) = \varepsilon_{\text{o}}[E_z^{\text{i}}(x,y,z=0^+,t) - E_z^{\text{t}}(x,y,z=0^-,t)]$$

$$= \varepsilon_{\text{o}}(1-n^{-2})E_{\text{p}}^{\text{i}}\sin\theta_{\text{B}}\exp[\text{i}(\omega/c)(x\sin\theta_{\text{B}} - ct)]. \quad (8)$$

Problem 2)

a) Incident beam: $\boldsymbol{k}^{\text{i}} = -(\omega/c)\hat{\boldsymbol{z}}$; $\boldsymbol{E}_{\text{o}}^{\text{i}} = E_{x\text{o}}^{\text{i}}\hat{\boldsymbol{x}}$; $\boldsymbol{H}_{\text{o}}^{\text{i}} = -(E_{x\text{o}}^{\text{i}}/Z_{\text{o}})\,\hat{\boldsymbol{y}}$.

$$\boldsymbol{E}^{\text{i}}(\boldsymbol{r},t) = E_{x\text{o}}^{\text{i}}\,\hat{\boldsymbol{x}}\exp[\text{i}(\omega/c)(-z-ct)]; \quad (1\text{a})$$

$$\boldsymbol{H}^{\text{i}}(\boldsymbol{r},t) = -(E_{x\text{o}}^{\text{i}}/Z_{\text{o}})\,\hat{\boldsymbol{y}}\exp[\text{i}(\omega/c)(-z-ct)]. \quad (1\text{b})$$

b) Reflected beam: $\boldsymbol{k}^{\text{r}} = (\omega/c)\hat{\boldsymbol{z}}$; $\boldsymbol{E}_{\text{o}}^{\text{r}} = E_{x\text{o}}^{\text{r}}\hat{\boldsymbol{x}}$; $\boldsymbol{H}_{\text{o}}^{\text{r}} = (E_{x\text{o}}^{\text{r}}/Z_{\text{o}})\,\hat{\boldsymbol{y}}$.

$$\boldsymbol{E}^{\text{r}}(\boldsymbol{r},t) = E_{x\text{o}}^{\text{r}}\,\hat{\boldsymbol{x}}\exp[\text{i}(\omega/c)(z-ct)]; \quad (2\text{a})$$

$$\boldsymbol{H}^{\text{r}}(\boldsymbol{r},t) = (E_{x\text{o}}^{\text{r}}/Z_{\text{o}})\,\hat{\boldsymbol{y}}\exp[\text{i}(\omega/c)(z-ct)]. \quad (2\text{b})$$

c) Beams inside dielectric: $\boldsymbol{k}^{\text{t}} = \pm(\omega/c)(n'+\text{i}n'')\,\hat{\boldsymbol{z}}$; $\boldsymbol{E}_{\text{o}}^{\text{t}} = \pm E_{x\text{o}}^{\text{t}}\hat{\boldsymbol{x}}$; $\boldsymbol{H}_{\text{o}}^{\text{t}} = (n'+\text{i}n'')(E_{x\text{o}}^{\text{t}}/Z_{\text{o}})\,\hat{\boldsymbol{y}}$. Note that we are dealing here with two plane-waves. The one propagating upward is given the plus sign, whereas that moving downward has the minus sign. The E-field amplitudes of these two beams have equal magnitudes and opposite signs, the reason being that, at the interface with the perfect conductor at $z=0$, the total tangential E-field must vanish. As usual, the H-field amplitude is derived from Maxwell's third equation, $\boldsymbol{k}\times\boldsymbol{E}_{\text{o}} = \mu_{\text{o}}\omega\boldsymbol{H}_{\text{o}}$. The total fields inside the dielectric layer are obtained by superposing these two plane-waves, as follows:

$$\boldsymbol{E}^{\text{t}}(\boldsymbol{r},t) = E_{x\text{o}}^{\text{t}}\,\hat{\boldsymbol{x}}\exp\{\text{i}(\omega/c)[(n'+\text{i}n'')z-ct]\} - E_{x\text{o}}^{\text{t}}\,\hat{\boldsymbol{x}}\exp\{\text{i}(\omega/c)[-(n'+\text{i}n'')z-ct]\}$$

$$= E_{x\text{o}}^{\text{t}}\,\hat{\boldsymbol{x}}[\exp(-n''\omega z/c)\exp(\text{i}n'\omega z/c) - \exp(n''\omega z/c)\exp(-\text{i}n'\omega z/c)]\exp(-\text{i}\omega t); \quad (3\text{a})$$

$$\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r},t)=(n'+\mathrm{i}n'')(E_{xo}^{\mathrm{t}}/Z_o)\,\hat{\boldsymbol{y}}\,[\exp(-n''\omega z/c)\exp(\mathrm{i}n'\omega z/c)$$
$$+\exp(n''\omega z/c)\exp(-\mathrm{i}n'\omega z/c)]\exp(-\mathrm{i}\omega t). \tag{3b}$$

d) Continuity of tangential E- and H-fields at $z=d$, the interface between free-space and the dielectric, yields

$$E_{xo}^{\mathrm{i}}\exp(-\mathrm{i}\omega d/c)+E_{xo}^{\mathrm{r}}\exp(\mathrm{i}\omega d/c)$$
$$=E_{xo}^{\mathrm{t}}[\exp(-n''\omega d/c)\exp(\mathrm{i}n'\omega d/c)-\exp(n''\omega d/c)\exp(-\mathrm{i}n'\omega d/c)]; \tag{4a}$$

$$-E_{xo}^{\mathrm{i}}\exp(-\mathrm{i}\omega d/c)+E_{xo}^{\mathrm{r}}\exp(\mathrm{i}\omega d/c)$$
$$=(n'+\mathrm{i}n'')E_{xo}^{\mathrm{t}}[\exp(-n''\omega d/c)\exp(\mathrm{i}n'\omega d/c)+\exp(n''\omega d/c)\exp(-\mathrm{i}n'\omega d/c)]. \tag{4b}$$

Subtracting Eq.(4b) from Eq.(4a) results in the cancellation of the term containing E_{xo}^{r}, yielding the following relation between the incident and transmitted E-field amplitudes:

$$\frac{E_{xo}^{\mathrm{t}}}{E_{xo}^{\mathrm{i}}}=\frac{2\exp(-\mathrm{i}\omega d/c)}{(1-n'-\mathrm{i}n'')\exp(-n''\omega d/c)\exp(\mathrm{i}n'\omega d/c)-(1+n'+\mathrm{i}n'')\exp(n''\omega d/c)\exp(-\mathrm{i}n'\omega d/c)}. \tag{5}$$

Substituting the above expression into Eq.(4a) then yields the ratio of reflected to incident E-field amplitudes as

$$\frac{E_{xo}^{\mathrm{r}}}{E_{xo}^{\mathrm{i}}}=\exp(-\mathrm{i}2\omega d/c)$$
$$\times\frac{(1+n'+\mathrm{i}n'')\exp(-n''\omega d/c)\exp(\mathrm{i}n'\omega d/c)-(1-n'-\mathrm{i}n'')\exp(n''\omega d/c)\exp(-\mathrm{i}n'\omega d/c)}{(1-n'-\mathrm{i}n'')\exp(-n''\omega d/c)\exp(\mathrm{i}n'\omega d/c)-(1+n'+\mathrm{i}n'')\exp(n''\omega d/c)\exp(-\mathrm{i}n'\omega d/c)}. \tag{6}$$

e) The time-averaged Poynting vector at $z=d^-$ is given by

$$<\boldsymbol{S}(x,y,z=d^-,t)>=\tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}^{\mathrm{t}}(x,y,z=d,t)\times\boldsymbol{H}^{\mathrm{t}*}(x,y,z=d,t)]$$
$$=\frac{|E_{xo}^{\mathrm{t}}|^2\hat{\boldsymbol{z}}}{2Z_o}\mathrm{Re}\{(n'-\mathrm{i}n'')[\exp(-n''\omega d/c)\exp(\mathrm{i}n'\omega d/c)-\exp(n''\omega d/c)\exp(-\mathrm{i}n'\omega d/c)]$$
$$\times[\exp(-n''\omega d/c)\exp(-\mathrm{i}n'\omega d/c)+\exp(n''\omega d/c)\exp(\mathrm{i}n'\omega d/c)]\}.$$
$$=\frac{|E_{xo}^{\mathrm{t}}|^2\hat{\boldsymbol{z}}}{2Z_o}\mathrm{Re}\{(n'-\mathrm{i}n'')[\exp(-2n''\omega d/c)-\exp(2n''\omega d/c)+\exp(\mathrm{i}2n'\omega d/c)-\exp(-\mathrm{i}2n'\omega d/c)]\}$$
$$=\frac{|E_{xo}^{\mathrm{t}}|^2\hat{\boldsymbol{z}}}{Z_o}\mathrm{Re}\{(n'-\mathrm{i}n'')[-\sinh(2n''\omega d/c)+\mathrm{i}\sin(2n'\omega d/c)]\}$$
$$=-\frac{|E_{xo}^{\mathrm{t}}|^2}{Z_o}[n'\sinh(2n''\omega d/c)-n''\sin(2n'\omega d/c)]\hat{\boldsymbol{z}}. \tag{7}$$

Problem 3)

a) $\rho_p = 0 \quad \rightarrow \quad \varepsilon_a\sqrt{\mu_b\varepsilon_b - (ck_x/\omega)^2} = \varepsilon_b\sqrt{\mu_a\varepsilon_a - (ck_x/\omega)^2}$ where $k_x = (\omega/c)\sqrt{\mu_a\varepsilon_a}\sin\theta_{Bp}$. Therefore,

$\varepsilon_a^2(\mu_b\varepsilon_b - \mu_a\varepsilon_a\sin^2\theta_{Bp}) = \varepsilon_b^2(\mu_a\varepsilon_a - \mu_a\varepsilon_a\sin^2\theta_{Bp}) \rightarrow \sin^2\theta_{Bp} = (\varepsilon_b/\mu_a)(\varepsilon_a\mu_b - \varepsilon_b\mu_a)/(\varepsilon_a^2 - \varepsilon_b^2)$

$\rightarrow \cos^2\theta_{Bp} = 1 - \sin^2\theta_{Bp} = (\varepsilon_a/\mu_a)(\varepsilon_a\mu_a - \varepsilon_b\mu_b)/(\varepsilon_a^2 - \varepsilon_b^2)$

$\rightarrow \tan^2\theta_{Bp} = (\varepsilon_b/\varepsilon_a)(\varepsilon_a\mu_b - \varepsilon_b\mu_a)/(\varepsilon_a\mu_a - \varepsilon_b\mu_b).$

b) $\rho_s = 0 \quad \rightarrow \quad \mu_a\sqrt{\mu_b\varepsilon_b - (ck_x/\omega)^2} = \mu_b\sqrt{\mu_a\varepsilon_a - (ck_x/\omega)^2}$ where $k_x = (\omega/c)\sqrt{\mu_a\varepsilon_a}\sin\theta_{Bs}$. Therefore,

$\mu_a^2(\mu_b\varepsilon_b - \mu_a\varepsilon_a\sin^2\theta_{Bs}) = \mu_b^2(\mu_a\varepsilon_a - \mu_a\varepsilon_a\sin^2\theta_{Bs}) \rightarrow \sin^2\theta_{Bs} = (\mu_b/\varepsilon_a)(\mu_a\varepsilon_b - \mu_b\varepsilon_a)/(\mu_a^2 - \mu_b^2)$

$\rightarrow \cos^2\theta_{Bs} = 1 - \sin^2\theta_{Bs} = (\mu_a/\varepsilon_a)(\varepsilon_a\mu_a - \varepsilon_b\mu_b)/(\mu_a^2 - \mu_b^2)$

$\rightarrow \tan^2\theta_{Bs} = -(\mu_b/\mu_a)(\varepsilon_a\mu_b - \varepsilon_b\mu_a)/(\varepsilon_a\mu_a - \varepsilon_b\mu_b).$

c) In the above expressions for $\tan^2\theta_{Bp}$ and $\tan^2\theta_{Bs}$, the second and third terms are identical. As for the first terms, the signs of ε_a and μ_a are generally the same, and so are the signs of ε_b and μ_b. Therefore, the signs of $\tan^2\theta_{Bp}$ and $\tan^2\theta_{Bs}$ are going to be opposite, that is, if one is positive, the other will be negative. Since tangent-squared needs to be positive, it will be *impossible* to have Brewster's angles for both p- and s-light.