Problem 1) a) $$\text{E-field energy} = \tfrac{1}{2}\epsilon_0(|E_{x0}|^2+|E_{y0}|^2)\iiint_{x\,y\,z=0}^{c\tau}\cos^2(k_0z-\omega t)\,dx\,dy\,dz$$

$$= \tfrac{1}{4}\epsilon_0(|E_{x0}|^2+|E_{y0}|^2)Ac\tau$$

$$\text{H-field energy} = \tfrac{1}{4}\mu_0(|H_{x0}|^2+|H_{y0}|^2)Ac\tau = \tfrac{1}{4}\frac{\mu_0}{Z_0^2}(|E_{y0}|^2+|E_{x0}|^2)Ac\tau$$

$$Z_0^2=\frac{\mu_0}{\epsilon_0} \rightarrow \quad = \tfrac{1}{4}\epsilon_0(|E_{x0}|^2+|E_{y0}|^2)Ac\tau$$

$$\text{Total energy} = \text{E-field energy} + \text{H-field energy} = \tfrac{1}{2}\epsilon_0(|E_{x0}|^2+|E_{y0}|^2)Ac\tau$$

b) $$\langle\vec{S}(\vec{r},t)\rangle = \tfrac{1}{2}\mathrm{Re}(E\times H^*) = \tfrac{1}{2}\mathrm{Re}\{(E_{x0}\hat{x}+E_{y0}\hat{y})\times(H^*_{x0}\hat{x}+H^*_{y0}\hat{y})\}$$

$$= \tfrac{1}{2}\mathrm{Re}\left\{(E_{x0}\hat{x}+E_{y0}\hat{y})\times\frac{1}{Z_0}(-E^*_{y0}\hat{x}+E^*_{x0}\hat{y})\right\} = \frac{1}{2Z_0}(E_{x0}E^*_{x0}+E_{y0}E^*_{y0})\hat{z}$$

$$= \frac{1}{2Z_0}(|E_{x0}|^2+|E_{y0}|^2)\hat{z}$$

Now, $\langle\vec{S}(\vec{r},t)\rangle$ is the rate of flow of energy per unit cross-sectional area of the pulse per unit time. To find the total pulse energy, we multiply $\langle\vec{S}\rangle$ with A (cross-sectional area) and τ (pulse duration):

$$\text{Total energy} = \langle S\rangle A\tau = \frac{1}{2Z_0}(|E_{x0}|^2+|E_{y0}|^2)A\tau.$$

Comparison with part (a) shows that the two expressions for total energy are identical because $\epsilon_0 c = \dfrac{\epsilon_0}{\sqrt{\mu_0\epsilon_0}} = \dfrac{1}{\sqrt{\mu_0/\epsilon_0}} = \dfrac{1}{Z_0}$ ✓

c) At normal incidence (i.e., $\theta=0$) we have: $r_p=r_s=\dfrac{1-n}{1+n}=\dfrac{1-(2+0.25i)}{1+(2+0.25i)}$

$$= \frac{-1-0.25i}{3+0.25i} \Rightarrow R=|r_p|^2=|r_s|^2=\frac{1+0.25^2}{3^2+0.25^2}=0.1172$$

The energy absorbed is the energy of the incident pulse minus that of the reflected pulse, that is, $\mathcal{E}_{\text{absorbed}}=(1-0.1172)\mathcal{E}_{\text{pulse}}=0.8828\,\mathcal{E}_{\text{pulse}}$.

d) The electromagnetic momentum of the pulse is $\mathcal{E}_{pulse}/c = \frac{1}{2}\epsilon_0(|E_{x_0}|^2+|E_{y_0}|^2)A\tau$. The momentum acquired by the material medium is the momentum of the incident pulse plus that of the reflected pulse, because it is the change of momentum (i.e., the differece between incident and reflected momenta) that is transferred to the absorbing medium. Thus: Mechanical momentum of the material medium $= (1+0.1172)\,\mathcal{E}_{pulse}/c =$
$0.5586\,\epsilon_0(|E_{x_0}|^2+|E_{y_0}|^2)A\tau.$

Problem 2) a) $$C(\omega)=\frac{\omega_p^2}{\omega_0^2-\omega^2-i\gamma\omega}=\frac{\omega_p^2(\omega_0^2-\omega^2)+i\omega_p^2\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}.$$

The denominator is always positive, and $\omega_p^2\gamma\omega \geq 0. \Rightarrow \mathrm{Im}\{C(\omega)\}\geq 0$.
In the numerator, the real part is $\omega_p^2(\omega_0^2-\omega^2)$, which is positive if $\omega_0>\omega$, zero if $\omega_0=\omega$, and negative if $\omega_0<\omega$. Therefore, $\mathrm{Re}\{C(\omega)\}$ could be positive, zero, or negative.

b) $$\chi(\omega)=\frac{3C(\omega)}{3-C(\omega)}=3\,\frac{c'+ic''}{3-c'-ic''}=3\,\frac{(c'+ic'')(3-c'+ic'')}{(3-c')^2+c''^2}=$$

$$3\,\frac{(3-c')c'-c''^2+ic'c''+ic''(3-c')}{(3-c')^2+c''^2}=3\,\frac{3c'-(c'^2+c''^2)+i3c''}{(3-c')^2+c''^2}\Rightarrow$$

$\mathrm{Im}\{\chi(\omega)\}=\dfrac{9c''}{(3-c')^2+c''^2}\geq 0$ (because denominator is positive and c'' in the numerator is ≥ 0.)

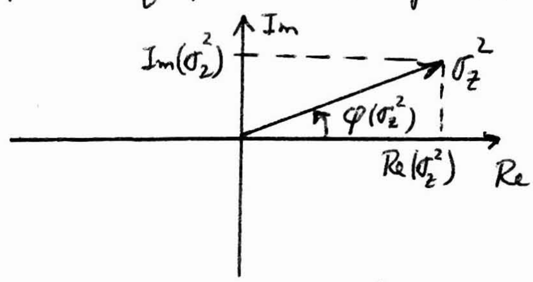
As for the real part of $\chi(\omega)$, the sign is determined by $3c'-(c'^2+c''^2)$. This will be negative when $c'<0$, and can be zero or positive for many combinations of $\omega, \omega_0, \omega_p, \gamma$. Therefore, $\mathrm{Re}\{\chi(\omega)\}$ can be positive, zero, or negative.

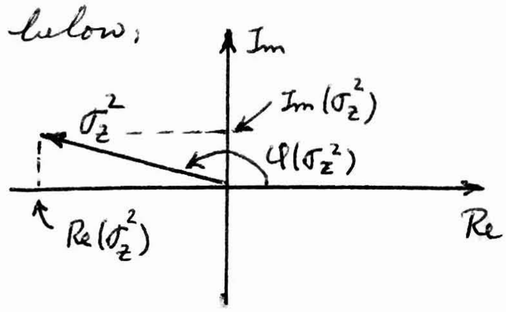
c) $\sigma_z^2 = 1+\chi(\omega)-\sigma_x^2-\sigma_y^2$ has real and imaginary parts given by

$$\mathrm{Re}\{\sigma_z^2\}=\mathrm{Re}\{\chi(\omega)\}+(1-\sigma_x^2-\sigma_y^2);\qquad \mathrm{Im}\{\sigma_z^2\}=\mathrm{Im}\{\chi(\omega)\}\geq 0.$$

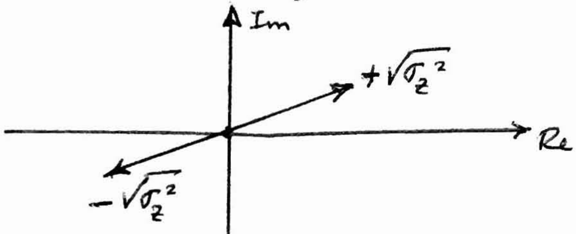
Clearly, the real part of σ_z^2 can be positive, zero, or negative, whereas the

Imaginary parts of σ_z^2 must always be non-negative. Possible locations of σ_z^2 in the complex-plane are, therefore, always in the upper half-plane. Examples are shown below:





The polar angle of σ_z^2, denoted by $\varphi(\sigma_z^2)$ in the above diagrams, is between zero and $180°$. This means that the polar angle of $\sigma_z = \sqrt{\sigma_z^2}$ is between $0°$ and $90°$. Therefore, $\sigma_z = +\sqrt{\sigma_z^2}$ is in the first quadrant, while $\sigma_z = -\sqrt{\sigma_z^2}$ is in the third quadrant, as shown below:



Clearly, one possible value of σ_z has $Re(\sigma_z) \geq 0$ and $Im(\sigma_z) \geq 0$, whereas the other possible value has $Re(\sigma_z) \leq 0$ and $Im(\sigma_z) \leq 0$. In general <u>both</u> values of σ_z are acceptable, unless the plane-wave happens to be in a semi-infinite medium where either $z \to \infty$ or $z \to -\infty$. Now the exponential function of the plane-wave is:

$$\exp[i(k_0\vec{\sigma}\cdot\vec{r} - \omega t)] = \exp[-k_0 Im(\sigma_z)z]\, \exp[ik_0(\sigma_x x + \sigma_y y + Re(\sigma_z)z - ct)].$$

Therefore, if $z \to +\infty$, the acceptable solution must have $Im(\sigma_z) \geq 0$; <u>in this case the "+ solution" is acceptable</u>. On the other hand, if $z \to -\infty$, the acceptable solution must have $Im(\sigma_z) \leq 0$; that is, the <u>"− solution" will be acceptable</u>. If the medium in which the plane-wave resides happens to have a finite thickness along the z-axis, then <u>both the "+ solution" and the "− solution" will coexist</u> within the medium.

Problem 3) a) $\vec{E}(\vec{r},t) = E_0 \hat{x} \exp\{ik_0[n(\omega)z - ct]\}$; $k_0 = \omega/c$

$Z_0 \vec{H}_0 = \vec{\sigma} \times \vec{E}_0 \Rightarrow Z_0 \vec{H}_0 = n(\omega) E_0 \hat{z} \times \hat{x} \Rightarrow \vec{H}_0 = \frac{n(\omega) E_0}{Z_0} \hat{y} \Rightarrow$

$$\vec{H}(\vec{r},t) = \frac{n(\omega) E_0}{Z_0} \hat{y} \exp\{ik_0[n(\omega)z - ct]\}. \checkmark$$

b) $n(\omega) = \sqrt{\epsilon(\omega)} \Rightarrow \epsilon(\omega) = n^2(\omega)$ ✓

$\epsilon(\omega) = 1 + \chi(\omega) \Rightarrow \chi(\omega) = n^2(\omega) - 1$ ✓

c) $\vec{P}(\vec{r},t) = \epsilon_0 \chi(\omega) E_0 \hat{x} \exp\{ik_0[n(\omega)z - ct]\}$

$= \epsilon_0 [n^2(\omega) - 1] E_0 \hat{x} \exp\{ik_0[n(\omega)z - ct]\}$ ✓

$\rho_{bound}(\vec{r},t) = -\vec{\nabla}\cdot\vec{P}(\vec{r},t) = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) = 0$ ✓

$\vec{J}_{bound} = \frac{\partial \vec{P}(\vec{r},t)}{\partial t} = -i\omega \epsilon_0 [n^2(\omega) - 1] E_0 \hat{x} \exp\{ik_0[n(\omega)z - ct]\}$

The actual $\vec{E}, \vec{H}, \vec{P}, \vec{J}_{bound}$ are, of course, the real parts of the above expressions.

Problem 4) a) $E_z(x, z=0^-, t) = E_z^{(incident)} + E_z^{(reflected)} \Rightarrow$

$E_z(x, z=0^-, t) = (r_p - 1) E_p \sin\theta \exp\{ik_0(\sin\theta\, x - ct)\}$; $k_0 = \omega/c$ ✓

reflected incident σ_x

b) $D_\perp(x, z=0, t) = \epsilon_0 E_z(x, z=0^-, t) = \epsilon_0 \epsilon(\omega) E_z(x, z=0^+, t) \Rightarrow$

$$E_z(x, z=0^+, t) = \frac{r_p - 1}{\epsilon(\omega)} E_p \sin\theta \exp\{ik_0(\sin\theta\, x - ct)\} \checkmark$$

c) $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow E_z(x, z=0^+, t) - E_z(x, z=0^-, t) = \frac{1}{\epsilon_0} \sigma_s(x, z=0, t) \Rightarrow$

$$\sigma_s(x, z=0, t) = \epsilon_0 \left[\frac{1}{\epsilon(\omega)} - 1\right] (r_p - 1) E_p \sin\theta \exp\{ik_0(\sin\theta\, x - ct)\} \checkmark$$

Problem 5) a) $n_g(\omega) = n(\omega) + \omega \frac{d}{d\omega}\left(1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2}\right)^{1/2} \Rightarrow$

$$n_g(\omega) = n(\omega) + \frac{\omega}{\cancel{2}n(\omega)} \frac{\cancel{2}\omega\omega_p^2}{(\omega^2-\omega_0^2)^2} = \frac{n^2(\omega)(\omega^2-\omega_0^2)^2 + \omega^2\omega_p^2}{n(\omega)(\omega^2-\omega_0^2)^2} \Rightarrow$$

$$n_g(\omega) = \frac{(\omega^2-\omega_0^2)^2 - \omega_p^2(\cancel{\omega^2}-\omega_0^2) + \cancel{\omega^2\omega_p^2}}{n(\omega)(\omega^2-\omega_0^2)^2} = \frac{1 + \left(\frac{\omega_0\omega_p}{\omega^2-\omega_0^2}\right)^2}{n(\omega)}$$

b) $n_g(\omega)\, n(\omega) = 1 + \left(\frac{\omega_0\omega_p}{\omega^2-\omega_0^2}\right)^2 > 1$ ✓

c) $n_g(\omega)n(\omega) > 1$ as shown in (b). Therefore, if $0 < n(\omega) < 1$, we must have $n_g(\omega) > 1$ to ensure that the product is greater than unity.

d) $n_g(\omega) > 1 \Rightarrow 1 + \left(\frac{\omega_0\omega_p}{\omega^2-\omega_0^2}\right)^2 > n(\omega) \Rightarrow 1 + 2\left(\frac{\omega_0\omega_p}{\omega^2-\omega_0^2}\right)^2 + \left(\frac{\omega_0\omega_p}{\omega^2-\omega_0^2}\right)^4 > n^2(\omega)$

$$\Rightarrow \cancel{1} + \frac{2\omega_0^2\omega_p^2}{(\omega_0^2-\omega^2)^2} + \frac{\omega_0^4\omega_p^4}{(\omega_0^2-\omega^2)^4} > \cancel{1} + \frac{\omega_p^2}{\omega_0^2-\omega^2} \Rightarrow$$

$$2\omega_0^2(\omega_0^2-\omega^2)^2 + \omega_0^4\omega_p^2 > (\omega_0^2-\omega^2)^3 \Rightarrow (\omega_0^2-\omega^2)^2(2\omega_0^2 - \omega_0^2 + \omega^2) + \omega_0^4\omega_p^2 > 0$$

$$\Rightarrow (\omega_0^2-\omega^2)^2(\omega_0^2+\omega^2) + \omega_0^4\omega_p^2 > 0$$

The last inequality is always valid. Therefore, $n_g(\omega) > 1$ is always valid. We have thus shown that whether $n(\omega) < 1$ or $n(\omega) > 1$, the Lorentz oscillator model always yields $n_g(\omega) > 1$.