1) a) To avoid reflection losses at the entrance and exit facets the angle of incidence $\theta$ must be equal to the Brewster angle $\theta_B$.

Therefore, $\theta = \theta_B = \tan^{-1} n = \tan^{-1}(1.5) \Rightarrow \theta = 56.31^\circ$ $\Rightarrow \theta' = 33.69^\circ$

Also, the reflection at the bottom must be total internal reflection, that is, $\phi > \phi_c$. Now $n \sin\phi_c = 1 \Rightarrow \phi_c = \sin^{-1}(1/n) = \sin^{-1}(2/3) = 41.81^\circ$.

Therefore, $\phi > 41.81^\circ$.

To determine the range of the prism angle $\psi$, use the triangle formed by a transmitted ray and the two sides of the prism. The sum of the angles of this triangle must be $180^\circ$, that is,

$$\psi + (90^\circ + \theta') + (90^\circ - \phi) = 180^\circ \Rightarrow \psi = \phi - \theta' \Rightarrow \psi > \phi_c - \theta' = 8.12^\circ.$$

Also, it is necessary that $\psi + (90^\circ + \theta') < 180^\circ \Rightarrow \psi < 56.31^\circ$, otherwise the beam inside the prism will not arrive at the bottom facet. (It will reach the top facet first.) Therefore, $8.12^\circ < \psi < 56.31^\circ$.

b) For s-light the Fresnel reflection coefficient is

$$r_s = \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}} = -0.38 \Rightarrow R_s = |r_s|^2 = 0.15$$

Incident optical power $= P_o$

Transmitted optical power $= P_o(1 - R_s)$ ← Note: Reflection loss is the same at entrance and exit facets.

Exiting optical power $= P_o(1-R_s)^2 = 0.85^2 P_o = 0.72 P_o$

Lost optical power due to reflections at the entrance and exit facets $= 28\%$.

c) Circularly-polarized beam is equal amounts p and s. The half that is p-polarized goes through without losses. The half that is s-polarized loses ~ 28%. Thus the total loss is ~14%

2) a) $\sigma_x'' = \sin\theta$ , $\sigma_y'' = 0$, $\sigma_z'' = \sqrt{\epsilon(\omega) - \sin^2\theta} = \left[(\epsilon_R(\omega) - \sin^2\theta) + i\epsilon_I(\omega)\right]^{1/2} \Rightarrow$

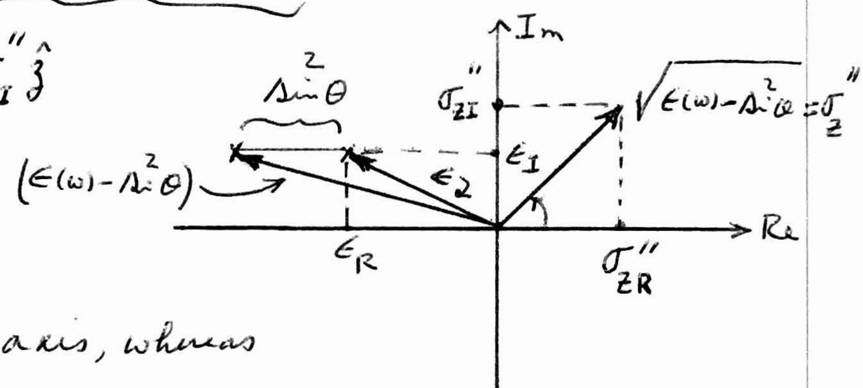
$\sigma_{zR}'' + i\sigma_{zI}''$

$$\vec{\sigma}'' = (\sin\theta)\,\hat{x} + \left[(\epsilon_R(\omega) - \sin^2\theta) + i\epsilon_I(\omega)\right]^{1/2}\hat{z}$$

b) When $\theta = 0°$, the x-component of $\vec{\sigma}''$ is zero (i.e., $\sin\theta) = 0$. Then $\vec{\sigma}''$ will only have a z-component, namely, $\sigma_z'' = \sqrt{\epsilon_R(\omega) + i\epsilon_I(\omega)}$. Therefore, both $\sigma_R''$ and $\sigma_I''$ are parallel to the z-axis, which makes $\vec{\sigma}_R'' \parallel \vec{\sigma}_I''$.

c) $\vec{\sigma}'' = (\sin\theta\,\hat{x} + \sigma_{zR}''\,\hat{z}) + i\sigma_{zI}''\,\hat{z}$

$$\Rightarrow \begin{cases} \vec{\sigma}_R'' = \sin\theta\,\hat{x} + \sigma_{zR}''\,\hat{z} \\ \vec{\sigma}_I'' = \sigma_{zI}''\,\hat{z} \end{cases}$$




Clearly, $\vec{\sigma}_I''$ is along the z-axis, whereas $\vec{\sigma}_R''$ has a component along $\hat{x}$ and another component along $\hat{z}$. The angle between $\vec{\sigma}_R''$ and $\vec{\sigma}_I''$ can therefore be anything between 0° and 90°.

d) When $\epsilon_I(\omega) = 0$, two possibilities exist; either $\epsilon_R(\omega) \geq \sin^2\theta$ or $\epsilon_R(\omega) < \sin^2\theta$.

Case I) $\epsilon_R(\omega) - \sin^2\theta \geq 0 \Rightarrow \vec{\sigma}_R'' = (\sin\theta)\,\hat{x} + \sqrt{\epsilon_R(\omega) - \sin^2\theta}\,\hat{z}$ ; $\vec{\sigma}_I'' = 0$

Case II) $\epsilon_R(\omega) - \sin^2\theta < 0 \Rightarrow \vec{\sigma}_R'' = (\sin\theta)\,\hat{x}$ ; $\vec{\sigma}_I'' = \sqrt{\sin^2\theta - \epsilon_R(\omega)}\,\hat{z}$

In Case II, note that $\vec{\sigma}_R'' \perp \vec{\sigma}_I''$. Therefore, $\vec{\sigma}_R''$ is orthogonal to $\vec{\sigma}_I''$ when $\epsilon_R(\omega) < 0$ or when $\sin^2\theta > \epsilon_R(\omega)$, provided that $\epsilon_I(\omega) = 0$ in either case.

3) $f(t) = \text{Real}\left\{\left[A_0 e^{i\phi_0} + A_1 e^{i(\phi_1+\Delta\omega t)} + A_2 e^{i(\phi_2-\Delta\omega t)}\right] e^{-i\omega_0 t}\right\}$

Defining $g(t) = A_0 e^{i\phi_0} + A_1 e^{i(\phi_1+\Delta\omega t)} + A_2 e^{i(\phi_2-\Delta\omega t)}$, we'll have

$$f(t) = \text{Real}\left\{g(t)\, e^{-i\omega_0 t}\right\}.$$

The entire function $g(t)$, therefore, rotates clockwise with frequency $\omega_0$ as a result of multiplication with $e^{-i\omega_0 t}$. But $g(t)$ also varies slowly in time because of the terms that contain $\Delta\omega t$. When the three vectors overlap, the magnitude of $g(t)$ will reach its maximum.

This happens when $\phi_1 + \Delta\omega t_0 = \phi_2 - \Delta\omega t_0 = \phi_0 \Rightarrow t_0 = \dfrac{\Delta\phi}{\Delta\omega}$.
The process is periodic, of course, repeating itself with a period $T$ where

$\Delta\omega T = 2\pi \Rightarrow T = \dfrac{2\pi}{\Delta\omega} \Rightarrow t_0 = mT + \dfrac{\Delta\phi}{\Delta\omega} \Rightarrow t_0 = \dfrac{2m\pi + \Delta\phi}{\Delta\omega}$,
where $m$ is any arbitrary integer.

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4) a)

$$E_x(\vec{r}_0, t) = \text{Real}\left\{\int_{\omega_1}^{\omega_2} E_0(\omega)\, e^{-\omega n_I(\omega) z_0/c}\, e^{i[\omega n_R(\omega) z_0/c - \omega t]}\, d\omega\right\}$$

Now, $\omega n_R(\omega)\frac{z_0}{c} - \omega t \cong \left(\omega_0 n_R(\omega_0)\frac{z_0}{c} - \omega_0 t\right) + \frac{d}{d\omega}\left[\omega n_R(\omega)\frac{z_0}{c} - \omega t\right]\Big|_{\omega=\omega_0}(\omega-\omega_0)$

$$= \left(\omega_0 n_R(\omega_0)\frac{z_0}{c} - \omega_0 t\right) + \left[n_R(\omega_0)\frac{z_0}{c} + \omega_0 n_R'(\omega_0)\frac{z_0}{c} - t\right](\omega-\omega_0) \leftarrow$$

Note: $n_R'(\omega_0) = \frac{d}{d\omega} n_R(\omega)\big|_{\omega_0}$

Therefore,

$$E_x(\vec{r}_0, t) \simeq \text{Real}\left\{e^{i\left(\omega_0 n_R(\omega_0)\frac{z_0}{c} - \omega_0 t\right)} \int_{\omega_1}^{\omega_2} \exp\left[-\omega n_I(\omega) z_0/c\right] E_0(\omega)\, e^{i\left[\left(n_R(\omega_0) + \omega_0 n_R'(\omega_0)\right)\frac{z_0}{c} - t\right](\omega-\omega_0)}\, d\omega\right\}$$

b) This is now similar to Problem 3 above, where the function under the integral sign varies slowly with time. The integral thus defines the envelope of the pulse, and the peak occurs where the phase-factor under the integral disappears. The peak thus occurs at time $t = t_0$ where we have:

$$\left[n_R(\omega_0) + \omega_0 n_R'(\omega_0)\right]\frac{z_0}{c} - t_0 = 0$$

c)

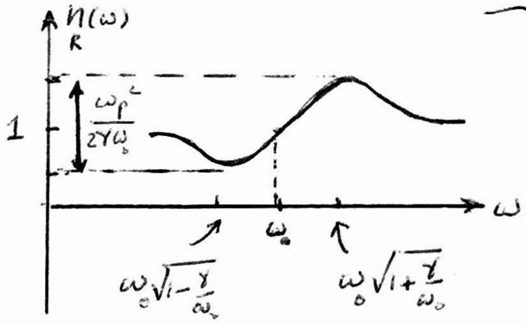
$$V_g = \frac{z_0}{t_0} = \frac{c}{n_R(\omega_0) + \omega_0 n_R'(\omega_0)}$$

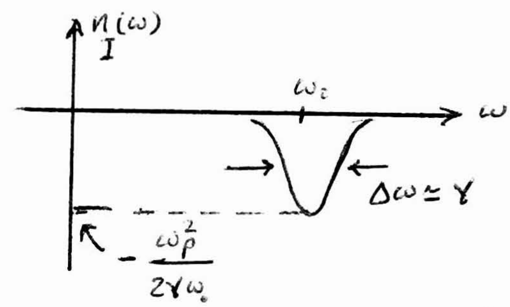
$n_R(\omega)$ participates in the phase-factor under the integral sign. Ultimately, of course, it's the phase-factor that determines the group velocity and, therefore, the function $n_R(\omega)$ and its derivative $n_R'(\omega)$ determine $V_g$. On the other hand, $n_I(\omega)$ does not affect the phase-factor; rather, it modifies the amplitude $E_0(\omega)$ of the various spectral components of the pulse. With increasing $z_0$, the attenuation coefficient $\exp[-\omega n_I(\omega) z_0/c]$, which modifies $E_0(\omega)$, becomes more prominent. Aside from an overall attenuation of the pulse, $n_I(\omega)$ also distorts the shape of the spectrum and, therefore, the shape of the pulse.

5) a) $n(\omega) = \sqrt{\epsilon(\omega)} = \sqrt{1+\chi_e(\omega)} \simeq 1 + \frac{1}{2}\chi_e(\omega) = 1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2 - \omega_0^2 + i\gamma\omega} \Rightarrow$

$$n_R(\omega) \simeq 1 + \frac{1}{2}\frac{\omega_p^2(\omega^2-\omega_0^2)}{(\omega^2-\omega_0^2)^2 + \gamma^2\omega^2} \quad ; \quad n_I(\omega) \simeq -\frac{1}{2}\frac{\gamma\omega\omega_p^2}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}$$

b)




c) $n_R(\omega_0) + \omega_0 n_R'(\omega_0) = 1 + \left(\frac{\omega_p}{\gamma}\right)^2 \Longrightarrow \boxed{V_g = \frac{c}{1+(\omega_p/\gamma)^2}}$ ← Note that $(\omega_p/\gamma)^2$ can be very large, and the pulse substantially slowed down.

For example, if $\omega_0 = 10^{15}$ rad/s, $\gamma = 10^{10}$ rad/s and $\omega_p = 10^{12}$ rad/sec, sub-nanosecond pulses will have $V_g \simeq 30$ Km/s.