1) a) Incident beam: $\vec{\sigma} = (0,0,-1)$. $Z_0 \vec{H}_0 = \vec{\sigma} \times \vec{E}_0 = -\hat{z} \times E_{0x}\hat{x} = -E_{0x}\hat{y}$

$$\langle S_z \rangle = \frac{1}{2}\,\mathrm{Re}(E_x H_y^*) = -\frac{1}{2Z_0}|E_{0x}|^2$$

← rate of flow of optical energy/Unit area in the incident beam (downward).

b)

Reflected beam: $\vec{\sigma}' = (0,0,+1)$, $Z_0 \vec{H}_0' = \vec{\sigma}' \times r\vec{E}_0 = \hat{z} \times rE_{0x}\hat{x} = rE_{0x}\hat{y}$

$$\langle S_z' \rangle = \frac{1}{2}\,\mathrm{Re}(E_x' H_y'^*) = \frac{1}{2}\,\mathrm{Re}\left(rE_{0x}\frac{r^*E_{0x}^*}{Z_0}\right) = \frac{1}{2Z_0}|r|^2|E_{0x}|^2 = \frac{R}{2Z_0}|E_{0x}|^2$$

Transmitted beam: $\vec{\sigma}'' = (0,0,-1)$, $Z_0 \vec{H}_0'' = \vec{\sigma}'' \times \tau\vec{E}_0 = -\hat{z} \times \tau E_{0x}\hat{x} = -\tau E_{0x}\hat{y}$

$$\langle S_z'' \rangle = \frac{1}{2}\,\mathrm{Re}(E_x'' H_y''^*) = -\frac{1}{2}\,\mathrm{Re}\left(\tau E_{0x}\frac{\tau^*E_{0x}^*}{Z_0}\right) = -\frac{1}{2Z_0}|\tau|^2|E_{0x}|^2 = -\frac{T}{2Z_0}|E_{0x}|^2$$

c) Since n_0 is real (i.e., slab is transparent), the fraction of incident energy that is reflected (i.e., R) plus the fraction that is transmitted (T) must equal unity. Therefore, $R + T = 1$

d) Momentum density $= \dfrac{\langle S_z \rangle}{c^2}\hat{z}$

In a short time Δt, the light travels a distance of $c\Delta t$ in free-space. With a unit-area cross-section, the corresponding volume is $c\Delta t$. Thus the momentum content of the volume is $\dfrac{\langle S_z \rangle}{c}\Delta t\,\hat{z}$. The force per unit area, $\vec{F}$, is given by:

$$\vec{F} = -\frac{\Delta \vec{p}}{\Delta t} = \frac{-1}{\Delta t}\left\{\langle S_z'' \rangle \frac{\Delta t}{c}\hat{z} + \langle S_z' \rangle \frac{\Delta t}{c}\hat{z} - \langle S_z \rangle \frac{\Delta t}{c}\hat{z}\right\} = -\frac{\langle S_z \rangle}{c}(T - R - 1)\hat{z}$$

$$= \frac{2R}{c}\langle S_z \rangle \hat{z} = -\frac{2R}{2cZ_0}|E_{x0}|^2\hat{z} \Rightarrow \vec{F} = -\epsilon_0 R |E_{x0}|^2 \hat{z}.$$

2) a) $\text{Energy density} = N\int_{E=0}^{E_0} \Delta\mathcal{E} = N\int_{E=0}^{E_0} \vec{E}\cdot\Delta\vec{p} = \int_{E=0}^{E_0} \vec{E}\cdot\Delta\vec{P} = \int_{E=0}^{E_0} E\,d(\epsilon_0\chi(0)E)$

$$= \epsilon_0\chi(0)\int_{E=0}^{E_0} E\,dE = \frac{1}{2}\epsilon_0\chi(0)E^2\Big|_{E=0}^{E_0} = \frac{1}{2}\epsilon_0\chi(0)E_0^2$$

In the above equation, N is the number of dipoles per unit volume, and $\vec{P} = N\vec{p}$, as usual. Note that $\chi(0) = \frac{Nq^2/m\epsilon_0}{\omega_0^2} = \frac{Nq^2}{\epsilon_0\alpha}$ and, therefore, Energy density $= \frac{1}{2}N(qE_0)^2/\alpha$. In the steady state $qE_0 = \alpha d$, where d is the length of the dipole; the force of the electric field on the negative charge, qE_0, is, therefore, fully balanced by the force of the spring (Constant of spring $= \alpha$) exerted on the negative charge when the length of the spring is d. Consequently: Energy density $= \frac{1}{2}N\alpha d^2$, which is the expression for the potential energy of N dipoles, each having a spring constant α, and stretched to length d.

b) Total E-field energy density = E-field energy density + Dipoles' energy density

$$= \frac{1}{2}\epsilon_0|E_0|^2 + \frac{1}{2}\epsilon_0\chi(0)|E_0|^2 = \frac{1}{2}\epsilon_0[1+\chi(0)]|E_0|^2 = \frac{1}{2}\epsilon_0\epsilon(0)|E_0|^2 \checkmark$$

(See Assignment 10, Prob. 6-h)

c) $\text{Energy density of dipoles} = \int_{E=0}^{E_0\cos(\omega t+\varphi_0)} \vec{E}\cdot d\vec{P} = \epsilon_0\chi(\omega)\int_{E=0}^{E_0\cos(\omega t+\varphi_0)} \vec{E}\cdot d\vec{E}$

$$= \frac{1}{2}\epsilon_0\chi(\omega)E_0^2\cos^2(\omega t+\varphi_0)$$

← Time-dependent energy density of the dipoles. The dipoles gain internal energy when elongated by the applied field. When the field reduces to zero, the dipoles shrink, returning their internal energy to the system in the form of radiation.

$$\text{Total E-field energy density} = \frac{1}{2}\epsilon_0 E_0^2\cos^2(\omega t+\varphi_0) + \frac{1}{2}\epsilon_0\chi(\omega)E_0^2\cos^2(\omega t+\varphi_0)$$

$$= \frac{1}{2}\epsilon_0\epsilon(\omega)E_0^2\cos^2(\omega t+\varphi_0) \Rightarrow \text{Time-averaged energy density} = \frac{1}{4}\epsilon_0\epsilon(\omega)|E_0|^2$$

d) Dipoles' Energy Density $= \int_{E=0}^{E(t)} \vec{E} \cdot d\vec{P} = \int_{t=0}^{t} \vec{E}(t') \cdot \frac{d\vec{P}(t')}{dt'}\, dt'$

$$= \epsilon_0 E_0^2 \int_0^t (\sin\omega_1 t' - \sin\omega_2 t')(\omega_1 \chi_1 \cos\omega_1 t' - \omega_2 \chi_2 \cos\omega_2 t')\,dt'$$

$$= \epsilon_0 E_0^2 \int_0^t [\omega_1\chi_1 \sin\omega_1 t' \cos\omega_1 t' + \omega_2\chi_2 \sin\omega_2 t' \cos\omega_2 t' - \omega_1\chi_1 \sin\omega_2 t' \cos\omega_1 t' - \omega_2 \chi_2 \sin\omega_1 t' \cos\omega_2 t']\,dt'$$

$$= \frac{1}{2}\epsilon_0 E_0^2 \int_0^t \{\omega_1\chi_1 \sin 2\omega_1 t' + \omega_2\chi_2 \sin 2\omega_2 t' - \omega_1\chi_1[\sin(\omega_1+\omega_2)t' + \sin(\omega_2-\omega_1)t'] - \omega_2\chi_2[\sin(\omega_1+\omega_2)t' + \sin(\omega_1-\omega_2)t']\}\,dt'$$

$$= \frac{1}{2}\epsilon_0 E_0^2 \left\{ -\frac{1}{2}\chi_1 \cos 2\omega_1 t' \Big|_0^t - \frac{1}{2}\chi_2 \cos 2\omega_2 t' \Big|_0^t + \frac{\omega_1\chi_1 + \omega_2\chi_2}{\omega_1+\omega_2} \cos(\omega_1+\omega_2)t' \Big|_0^t - \frac{\omega_2\chi_2 - \omega_1\chi_1}{\omega_2-\omega_1} \cos(\omega_2-\omega_1)t' \Big|_0^t \right\}$$

$$= \frac{1}{2}\epsilon_0 E_0^2 \left\{ -\frac{1}{2}\chi_1 \cos 2\omega_1 t + \frac{1}{2}\chi_1 - \frac{1}{2}\chi_2 \cos 2\omega_2 t + \frac{1}{2}\chi_2 + \frac{\omega_1\chi_1+\omega_2\chi_2}{\omega_1+\omega_2}[\cos(\omega_1+\omega_2)t - 1] - \frac{\omega_2\chi_2 - \omega_1\chi_1}{\omega_2-\omega_1}[\cos(\omega_2-\omega_1)t - 1] \right\}$$

Note that $\cos 2\omega_1 t = \cos(2m-1)\Delta\omega t$ has an integer number of periods between $t=0$ and $t = T = 2\pi/\Delta\omega$; therefore, time-averaging over the interval $(0,T)$ eliminates the term containing $\cos 2\omega_1 t$. Similarly, $\cos 2\omega_2 t = \cos(2m+1)\Delta\omega t$ averages out to zero. The same is true of $\cos(\omega_1+\omega_2)t = \cos 2m\Delta\omega t$ and $\cos(\omega_1-\omega_2)t = \cos\Delta\omega t$. Consequently,

Dipoles' Energy Density, time averaged over the interval $(0,T)$ =

$$\frac{1}{2}\epsilon_0 E_0^2 \left(\frac{1}{2}\chi_1 + \frac{1}{2}\chi_2 - \frac{\omega_1\chi_1+\omega_2\chi_2}{\omega_1+\omega_2} + \frac{\omega_2\chi_2 - \omega_1\chi_1}{\omega_2-\omega_1}\right) =$$

$$\frac{1}{2}\epsilon_0 E_0^2 \left\{ \frac{1}{2}\chi(\omega_1) + \frac{1}{2}\chi(\omega_2) - \frac{(m-\frac{1}{2})\cancel{\Delta\omega}\chi(\omega_1) + (m+\frac{1}{2})\cancel{\Delta\omega}\chi(\omega_2)}{2m\cancel{\Delta\omega}} + \frac{(m+\frac{1}{2})\Delta\omega\chi(\omega_2) - (m-\frac{1}{2})\Delta\omega\chi(\omega_1)}{\Delta\omega} \right\}$$

$$= \frac{1}{2}\epsilon_0 E_0^2 \left\{ \frac{\chi(\omega_1)-\chi(\omega_2)}{4m} + \frac{\chi(\omega_1)+\chi(\omega_2)}{2} + (m\Delta\omega)\frac{\chi(\omega_2)-\chi(\omega_1)}{\Delta\omega} \right\}$$

In the limit when $\Delta\omega \to 0$, we'll have $m \gg 1$, and the first term in the above expression may be ignored. Denoting by ω_0 the central frequency $m\Delta\omega = \frac{1}{2}(\omega_1+\omega_2)$, one writes

$$\langle \text{Dipoles' Energy Density} \rangle \simeq \frac{1}{2}\epsilon_0 E_0^2 \left[\chi(\omega_0) + \omega_0 \frac{d\chi(\omega)}{d\omega}\Big|_{\omega=\omega_0} \right]$$

The next step involves the calculation of time-averaged E-field energy density over the interval $(0, T)$:

Time-averaged E-field energy density $= \frac{1}{2}\epsilon_0 \langle E^2(t) \rangle$

$$= \frac{\epsilon_0 E_0^2}{2T}\int_0^T (\sin\omega_1 t - \sin\omega_2 t)^2 dt = \frac{\epsilon_0 E_0^2}{2T}\int_0^T (\sin^2\omega_1 t + \sin^2\omega_2 t - 2\sin\omega_1 t \sin\omega_2 t)\,dt$$

$$= \frac{\epsilon_0 E_0^2}{2T}\int_0^T \left\{ \frac{1-\cos 2\omega_1 t}{2} + \frac{1-\cos 2\omega_2 t}{2} + \cos(\omega_1+\omega_2)t - \cos(\omega_1-\omega_2)t \right\} dt = \frac{1}{2}\epsilon_0 E_0^2$$

Therefore, $\langle$ E-field energy density + Dipoles' energy density $\rangle$ =

$$\frac{1}{2}\epsilon_0 E_0^2 \left[1 + \chi(\omega_0) + \omega_0 \chi'(\omega_0) \right] = \frac{1}{2}\epsilon_0 \left[\epsilon(\omega_0) + \omega_0 \epsilon'(\omega_0) \right] E_0^2 = \frac{1}{2}\epsilon_0 \frac{d[\omega\epsilon(\omega)]}{d\omega} E_0^2$$

Noting that the time-averaged E-field intensity over the beat period T is $\langle E^2(t) \rangle = E_0^2$, we have: $\langle$Total energy density associated with E-field$\rangle$

$$= \frac{1}{2}\epsilon_0 \frac{d[\omega\epsilon(\omega)]}{d\omega} \langle E^2(t) \rangle$$

As a check on the above result, consider a plane-wave (quasi-monochromatic) having E-field amplitude $\hat{E}_0$, H-field amplitude $\hat{H}_0 = \frac{n(\omega)}{Z_0}\hat{E}_0$, propagating in a transparent, dispersive medium of refractive index $n(\omega) = \sqrt{\epsilon(\omega)} = \sqrt{1+\chi(\omega)}$. Then:

$$\langle \text{E-field energy density} \rangle + \langle \text{H-field energy density} \rangle = \frac{1}{4}\epsilon_0(\epsilon + \omega\epsilon')\hat{E}_0^2 + \frac{1}{4}\mu_0 \hat{H}_0^2$$

$$= \frac{1}{4}\epsilon_0(\epsilon + \omega\epsilon' + n^2)\hat{E}_0^2 = \frac{1}{2}\epsilon_0 n(\omega)\left[n(\omega) + \frac{\omega\epsilon'(\omega)}{2n(\omega)}\right]\hat{E}_0^2 = \frac{n(\omega)}{2Z_0 c}\left[n(\omega) + \omega n'(\omega)\right]\hat{E}_0^2 = \frac{\langle S_z \rangle}{v_g}.$$

In other words, the average energy density multiplied by group velocity v_g yields the Poynting component $\langle S_z \rangle$.

3a, b) Incident beam: $\sigma^{(i)} = (n_0 \sin\theta,\ 0,\ -n_0 \cos\theta)$

$$Z_0 H_0^{(i)} = \sigma^{(i)} \times E_0^{(i)} = n_0(\sin\theta\,\hat{x} - \cos\theta\,\hat{z}) \times E_{oy}^{(i)}\,\hat{y} \Rightarrow H_0^{(i)} = \frac{n_0}{Z_0}(\cos\theta\,\hat{x} + \sin\theta\,\hat{z}) E_{oy}^{(i)}$$

Reflected beam: $\sigma^{(r)} = (n_0 \sin\theta,\ 0,\ n_0 \cos\theta)$

$$Z_0 H_0^{(r)} = \sigma^{(r)} \times E_0^{(r)} = n_0(\sin\theta\,\hat{x} + \cos\theta\,\hat{z}) \times E_{oy}^{(r)}\,\hat{y} \Rightarrow H_0^{(r)} = \frac{n_0}{Z_0}(-\cos\theta\,\hat{x} + \sin\theta\,\hat{z})\, r E_{oy}^{(i)}$$

Beam $\underline{a}$ in the gap: $\sigma^{(a)} = (n_0 \sin\theta,\ 0,\ -i\sqrt{n_0^2 \sin^2\theta - 1}\,)$ ← Note: $\theta > \theta_c$ means that $n_0 \sin\theta > 1$

$$Z_0 H_0^{(a)} = \sigma^{(a)} \times E_0^{(a)} = (n_0 \sin\theta\,\hat{x} - i\sqrt{n_0^2 \sin^2\theta - 1}\,\hat{z}) \times E_{oy}^{(a)}\,\hat{y} \Rightarrow$$

$$H_0^{(a)} = \frac{1}{Z_0}\left(i\sqrt{n_0^2 \sin^2\theta - 1}\,\hat{x} + n_0 \sin\theta\,\hat{z}\right) a E_{oy}^{(i)}$$

$-i$ in front of $\sigma_z^{(a)}$ means the beam decays along the negative z-axis.

Beam $\underline{b}$ in the gap: $\sigma^{(b)} = (n_0 \sin\theta,\ 0,\ +i\sqrt{n_0^2 \sin^2\theta - 1}\,)$

$$Z_0 H_0^{(b)} = \sigma^{(b)} \times E_0^{(b)} = (n_0 \sin\theta\,\hat{x} + i\sqrt{n_0^2 \sin^2\theta - 1}\,\hat{z}) \times E_{oy}^{(b)}\,\hat{y} \Rightarrow$$

$$H_0^{(b)} = \frac{1}{Z_0}\left(-i\sqrt{n_0^2 \sin^2\theta - 1}\,\hat{x} + n_0 \sin\theta\,\hat{z}\right) b E_{oy}^{(i)}$$

Transmitted beam: $\sigma^{(t)} = (n_0 \sin\theta,\ 0,\ -n_0 \cos\theta)$

$$Z_0 H_0^{(t)} = \sigma^{(t)} \times E_0^{(t)} = (n_0 \sin\theta\,\hat{x} - n_0 \cos\theta\,\hat{z}) \times E_{oy}^{(t)}\,\hat{y} \Rightarrow H_0^{(t)} = \frac{n_0}{Z_0}(\cos\theta\,\hat{x} + \sin\theta\,\hat{z})\, \tau E_{oy}^{(i)}$$

c) Continuity equations at $z = 0$:

E_y-field: $E_{oy}^{(i)} + E_{oy}^{(r)} = E_{oy}^{(a)} + E_{oy}^{(b)} \Rightarrow 1 + r = a + b$

H_x-field: $\frac{n_0}{Z_0}\cos\theta\, E_{oy}^{(i)} - \frac{n_0}{Z_0}\cos\theta\, E_{oy}^{(r)} = \frac{i\sqrt{n_0^2 \sin^2\theta - 1}}{Z_0} E_{oy}^{(a)} - \frac{i\sqrt{n_0^2 \sin^2\theta - 1}}{Z_0} E_{oy}^{(b)}$

$$\Rightarrow 1 - r = \frac{i\sqrt{n_0^2 \sin^2\theta - 1}}{n_0 \cos\theta}(a - b)$$

Continuity equations at $z=-d$:

E_y-field: $E_{oy}^{(a)} e^{-ik_o\sigma_z^{(a)}d} + E_{oy}^{(b)} e^{-ik_o\sigma_z^{(b)}d} = E_{oy}^{(t)} e^{-ik_o\sigma_z^{(t)}d} \Rightarrow$

$$a e^{-k_o d\sqrt{n_o^2\Delta^2\theta-1}} + b e^{+k_o d\sqrt{n_o^2\Delta^2\theta-1}} = \tau e^{ik_o n_o d\cos\theta}$$

H_x-field: $\frac{i\sqrt{n_o^2\Delta^2\theta-1}}{Z_o} E_{oy}^{(a)} e^{-k_o d\sqrt{n_o^2\Delta^2\theta-1}} - \frac{i\sqrt{n_o^2\Delta^2\theta-1}}{Z_o} E_{oy}^{(b)} e^{+k_o d\sqrt{n_o^2\Delta^2\theta-1}}$

$$= \frac{n_o\cos\theta}{Z_o} E_{oy}^{(t)} e^{ik_o n_o d\cos\theta}$$

$$\Rightarrow \frac{i\sqrt{n_o^2\Delta^2\theta-1}}{n_o\cos\theta}\left(a e^{-k_o d\sqrt{n_o^2\Delta^2\theta-1}} - b e^{+k_o d\sqrt{n_o^2\Delta^2\theta-1}}\right) = \tau e^{ik_o n_o d\cos\theta}$$

d) Let $c = \frac{i\sqrt{n_o^2\Delta^2\theta-1}}{n_o\cos\theta}$. Then the first two equations can be solved for $\underline{a}$ and $\underline{b}$ in terms of $\underline{r}$, as follows:

$$\begin{cases} a+b = 1+r \\ a-b = \frac{1}{c}(1-r) \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2}(1+\frac{1}{c}) + \frac{r}{2}(1-\frac{1}{c}) \\ b = \frac{1}{2}(1-\frac{1}{c}) + \frac{r}{2}(1+\frac{1}{c}) \end{cases}$$

The 3rd and 4th equations yield a relation between $\underline{a}$ and $\underline{b}$ when their left-hand-sides are set equal to each other:

$$a e^{-k_o d\sqrt{\cdots}} + b e^{+k_o d\sqrt{\cdots}} = C\left(a e^{-k_o d\sqrt{\cdots}} - b e^{+k_o d\sqrt{\cdots}}\right) \Rightarrow$$

$$a e^{-k_o d\sqrt{\cdots}}(C-1) = b e^{+k_o d\sqrt{\cdots}}(C+1) \Rightarrow$$

$$\left[\frac{C+1}{\cancel{2C}} + \frac{r(C-1)}{\cancel{2C}}\right] e^{-2k_o d\sqrt{n_o^2\Delta^2\theta-1}}(C-1) = \left[\frac{C-1}{\cancel{2C}} + \frac{r(C+1)}{\cancel{2C}}\right](C+1) \Rightarrow$$

$$\left[(C^2-1) + r(C-1)^2\right] e^{-2k_o d\sqrt{n_o^2\Delta^2\theta-1}} = (C^2-1) + r(C+1)^2 \Rightarrow$$

$$(C^2-1)(1-e^{-2k_o d\sqrt{\cdots}}) = r\left[(C-1)^2 e^{-2k_o d\sqrt{}} - (C+1)^2\right] \Rightarrow$$

$$\boxed{r = \frac{1 - \exp(-2k_0 d\sqrt{n_0^2\sin^2\theta - 1}\,)}{\left(\frac{c-1}{c+1}\right)\exp(-2k_0 d\sqrt{n_0^2\sin^2\theta-1}\,) - \left(\frac{c+1}{c-1}\right)}} \quad \leftarrow k_0 = \frac{2\pi}{\lambda_0}$$

In the above equation $\frac{c-1}{c+1} = \frac{i\sqrt{n_0^2\sin^2\theta-1} - n_0\cos\theta}{i\sqrt{n_0^2\sin^2\theta-1} + n_0\cos\theta}$ is the Fresnel reflection coefficient for s-polarized light in the limit when $d \to \infty$. Once r is determined, one can substitute it in the preceding equations for $\underline{a}$ and $\underline{b}$ to determine these coefficients. Finally, $\underline{a}$ and $\underline{b}$ can be put into either the 3rd or the 4th continuity equation to determine τ.

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4) a) $\vec{E}_1(z,t) = E_{0x}\cos(k_1 z - \omega_1 t + \phi_1)\,\hat{x}$

$$\vec{H}_1(z,t) = \frac{E_{0x}}{Z_0}\cos(k_1 z - \omega_1 t + \phi_1)\,\hat{y}$$

$$\vec{E}_2(z,t) = E_{0x}\cos(k_2 z + \omega_2 t - \phi_2)\,\hat{x}$$

$$\vec{H}_2(z,t) = \frac{-E_{0x}}{Z_0}\cos(k_2 z + \omega_2 t - \phi_2)\,\hat{y}$$

b) $\vec{E}(z,t) = \vec{E}_1 + \vec{E}_2 = E_{0x}\,\hat{x}\,\{\cos(k_1 z - \omega_1 t + \phi_1) + \cos(k_2 z + \omega_2 t - \phi_2)\}$

$$= 2E_{0x}\,\hat{x}\cos\left[\tfrac{1}{2}(k_1+k_2)z + \tfrac{1}{2}(\omega_2-\omega_1)t + \tfrac{1}{2}(\phi_1-\phi_2)\right]\cos\left[\tfrac{1}{2}(k_1-k_2)z - \tfrac{1}{2}(\omega_1+\omega_2)t + \tfrac{1}{2}(\phi_1+\phi_2)\right]$$

$$= 2E_{0x}\,\hat{x}\cos\left[\frac{\omega_0 z}{c} + \frac{1}{2}\Delta\omega\, t + \frac{\phi_1-\phi_2}{2}\right]\cos\left[\frac{\Delta\omega z}{2c} + \omega_0 t - \frac{\phi_1+\phi_2}{2}\right]$$

$$\vec{H}(z,t) = \vec{H}_1 + \vec{H}_2 = \frac{E_{0x}}{Z_0}\,\hat{y}\,\{\cos(k_1 z - \omega_1 t + \phi_1) - \cos(k_2 z + \omega_2 t - \phi_2)\}$$

$$= +2\frac{E_{0x}}{Z_0}\,\hat{y}\sin\left[\frac{\omega_0 z}{c} + \frac{1}{2}\Delta\omega\, t + \frac{\phi_1-\phi_2}{2}\right]\sin\left[\frac{\Delta\omega z}{2c} + \omega_0 t - \frac{\phi_1+\phi_2}{2}\right]$$

c) E-field energy density $= \frac{1}{2}\epsilon_0 |E|^2 =$

$$2\epsilon_0 E_{0x}^2 \cos^2\left(\frac{\omega_0 z}{c} + \frac{1}{2}\Delta\omega t + \frac{\varphi_1 - \varphi_2}{2}\right)\cos^2\left(\frac{\Delta\omega z}{2c} + \omega_0 t - \frac{\varphi_1 + \varphi_2}{2}\right)$$

The time-averaged energy density over one period of rapid oscillation (i.e., $T = 2\pi/\omega_0$) may be obtained by ignoring the envelope fluctuations, as the envelope oscillates with the much lower frequency of $\Delta\omega$. Therefore,

$$\langle \text{E-field energy density} \rangle \simeq \epsilon_0 E_{0x}^2 \cos^2\left[\frac{\omega_0}{c}\left(z + \frac{c\Delta\omega}{2\omega_0}t\right) + \frac{\varphi_1 - \varphi_2}{2}\right]$$

Similarly,

$$\langle \text{H-field energy density} \rangle = \frac{1}{2}\mu_0 \langle H^2 \rangle \simeq \epsilon_0 E_{0x}^2 \sin^2\left[\frac{\omega_0}{c}\left(z + \frac{c\Delta\omega}{2\omega_0}t\right) + \frac{\varphi_1 - \varphi_2}{2}\right]$$

The energy densities have a period of $\lambda_0/2$ along the z-axis, are shifted (relative to each other) by $\lambda_0/4$, and travel in the same direction with a speed $v = \frac{c\Delta\omega}{2\omega_0}$. The direction of travel depends on the sign of $\Delta\omega = \omega_2 - \omega_1$; in general, the fringes travel in the direction of the beam that has the higher frequency. The <u>sum</u> of the two (time-averaged) energy densities, however, is constant and stationary.

d) $\vec{S} = \vec{E} \times \vec{H} = \frac{E_{0x}^2}{Z_0}\hat{z} \sin\left[\frac{2\omega_0}{c}\left(z + \frac{c\Delta\omega}{2\omega_0}t\right) + \varphi_1 - \varphi_2\right] \sin\left[\frac{\Delta\omega z}{c} + 2\omega_0 t - (\varphi_1 + \varphi_2)\right]$

The rapid oscillations of the second sinusoid (frequency $= 2\omega_0$) yield a time-averaged value of zero for $\vec{S}$. Therefore, $\langle S_z \rangle = 0$; the energy does <u>not</u> flow in either direction.

A particularly interesting case occurs when $\Delta\omega = 0$. Here the fringes are stationary, and the Poynting vector $\vec{S}(r, t)$ shows how the energy is exchanged between the $\vec{E}$- and $\vec{H}$-fields.