

1) a) Incident beam:  $\vec{\sigma} = (0, 0, -1)$ ,  $\vec{z}_0 \vec{H}_0 = \vec{\sigma} \times \vec{E}_0 = -\hat{z} \times E_{0x} \hat{x} = -E_{0x} \hat{y}$

$$\langle S_z \rangle = \frac{1}{2} \operatorname{Re}(E_x H_y^*) = -\frac{1}{2z_0} |E_{0x}|^2 \leftarrow \text{rate of flow of optical energy/Unit area in the incident beam (downward).}$$

b)

Reflected beam:  $\vec{\sigma}' = (0, 0, +1)$ ,  $\vec{z}_0 \vec{H}'_0 = \vec{\sigma}' \times r \vec{E}_0 = \hat{z} \times r E_{0x} \hat{x} = r E_{0x} \hat{y}$

$$\langle S'_z \rangle = \frac{1}{2} \operatorname{Re}(E'_x H'^*_y) = \frac{1}{2} \operatorname{Re}(r E_{0x} \frac{r^* E_{0x}^*}{z_0}) = \frac{1}{2z_0} |r|^2 |E_{0x}|^2 = \frac{R}{2z_0} |E_{0x}|^2$$

Transmitted beam:  $\vec{\sigma}'' = (0, 0, -1)$ ,  $\vec{z}_0 \vec{H}''_0 = \vec{\sigma}'' \times \tau \vec{E}_0 = -\hat{z} \times \tau E_{0x} \hat{x} = -\tau E_{0x} \hat{y}$

$$\langle S''_z \rangle = \frac{1}{2} \operatorname{Re}(E''_x H''^*_y) = -\frac{1}{2} \operatorname{Re}(\tau E_{0x} \frac{\tau^* E_{0x}^*}{z_0}) = -\frac{1}{2z_0} |\tau|^2 |E_{0x}|^2 = -\frac{T}{2z_0} |E_{0x}|^2$$

c) Since  $n_0$  is real (i.e., slab is transparent), the fraction of incident energy that is reflected (i.e.,  $R$ ) plus the fraction that is transmitted<sup>(T)</sup> must equal unity. Therefore,  $R + T = 1$

d) Momentum density =  $\frac{\langle S_z \rangle}{c^2} \hat{z}$

In a short time  $\Delta t$ , the light travels a distance of  $c \Delta t$  in free-space.

With a unit-area cross-section, the corresponding volume is  $c \Delta t$ .

Thus the momentum content of the volume is  $\frac{\langle S_z \rangle}{c} \Delta t \hat{z}$ . The force

per unit area,  $\vec{F}$ , is given by:

$$\vec{F} = -\frac{\Delta \vec{p}}{\Delta t} = -\frac{1}{\Delta t} \left\{ \langle S''_z \rangle \frac{\Delta t}{c} \hat{z} + \langle S'_z \rangle \frac{\Delta t}{c} \hat{z} - \langle S_z \rangle \frac{\Delta t}{c} \hat{z} \right\} = -\frac{\langle S_z \rangle}{c} (T - R - 1) \hat{z}$$

$$= \frac{2R}{c} \langle S_z \rangle \hat{z} = -\frac{2R}{2cz_0} |E_{0x}|^2 \hat{z} \Rightarrow \vec{F} = -\epsilon_0 R |E_{0x}|^2 \hat{z}$$



$$2) a) \text{ Energy density} = N \int_{E=0}^{E_0} \Delta \mathcal{E} = N \int_{E=0}^{E_0} \vec{E}_0 \cdot \Delta \vec{P} = \int_{E=0}^{E_0} \vec{E}_0 \cdot \Delta \vec{P} = \int_{E=0}^{E_0} E d(\epsilon_0 \chi(\omega) E)$$

$$= \epsilon_0 \chi(\omega) \int_{E=0}^{E_0} E dE = \frac{1}{2} \epsilon_0 \chi(\omega) E^2 \Big|_{E=0}^{E_0} = \frac{1}{2} \epsilon_0 \chi(\omega) E_0^2$$

In the above equation,  $N$  is the number of dipoles per unit volume, and  $\vec{P} = N\vec{p}$ , as usual. Note that  $\chi(\omega) = \frac{Nq^2/m\epsilon_0}{\omega_0^2} = \frac{Nq^2}{\epsilon_0\alpha}$  and, therefore, Energy density  $= \frac{1}{2} N(qE_0)^2/\alpha$ . In the steady state  $qE_0 = \alpha d$ , where  $d$  is the length of the dipole; the force of the electric field on the negative charge,  $qE_0$ , is, therefore, fully balanced by the force of the spring (constant of spring  $= \alpha$ ) exerted on the negative charge when the length of the spring is  $d$ . Consequently: Energy density  $= \frac{1}{2} N \alpha d^2$ , which is the expression for the potential energy of  $N$  dipoles, each having a spring constant  $\alpha$ , and stretched to length  $d$ .

$$b) \text{ Total E-field energy density} = \text{E-field energy density} + \text{Dipoles' energy density}$$

$$= \frac{1}{2} \epsilon_0 |E_0|^2 + \frac{1}{2} \epsilon_0 \chi(\omega) |E_0|^2 = \frac{1}{2} \epsilon_0 [1 + \chi(\omega)] |E_0|^2 = \frac{1}{2} \epsilon_0 \epsilon(\omega) |E_0|^2 \checkmark$$

(See Assignment 10, Prob. 6-h)

$$c) \text{ Energy density of dipoles} = \int_{E=0}^{E_0 \cos(\omega t + \phi_0)} \vec{E} \cdot d\vec{P} = \epsilon_0 \chi(\omega) \int_{E=0}^{E_0 \cos(\omega t + \phi_0)} \vec{E} \cdot d\vec{E}$$

$$= \frac{1}{2} \epsilon_0 \chi(\omega) E_0^2 \cos^2(\omega t + \phi_0)$$

← Time-dependent energy density of the dipoles. The dipoles gain internal energy when elongated by the applied field. When the field reduces to zero, the dipoles shrink, returning their internal energy to the system in the form of radiation.

$$\text{Total E-field energy density} = \frac{1}{2} \epsilon_0 E_0^2 \cos^2(\omega t + \phi_0) + \frac{1}{2} \epsilon_0 \chi(\omega) E_0^2 \cos^2(\omega t + \phi_0)$$

$$= \frac{1}{2} \epsilon_0 \epsilon(\omega) E_0^2 \cos^2(\omega t + \phi_0) \Rightarrow \text{Time-averaged energy density} = \frac{1}{4} \epsilon_0 \epsilon(\omega) |E_0|^2$$



$$\begin{aligned}
 \text{d) Dipoles' Energy Density} &= \int_{E=0}^{E(t)} \vec{E} \cdot d\vec{P} = \int_{t=0}^t \vec{E}(t') \cdot \frac{d\vec{P}(t')}{dt'} dt' \\
 &= \epsilon_0 E_0^2 \int_0^t (\sin \omega_1 t' - \sin \omega_2 t') (\omega_1 x_1 \cos \omega_1 t' - \omega_2 x_2 \cos \omega_2 t') dt' \\
 &= \epsilon_0 E_0^2 \int_0^t \left[ \omega_1 x_1 \sin \omega_1 t' \cos \omega_1 t' + \omega_2 x_2 \sin \omega_2 t' \cos \omega_2 t' - \omega_1 x_1 \sin \omega_2 t' \cos \omega_1 t' \right. \\
 &\quad \left. - \omega_2 x_2 \sin \omega_1 t' \cos \omega_2 t' \right] dt' \\
 &= \frac{1}{2} \epsilon_0 E_0^2 \int_0^t \left\{ \omega_1 x_1 \sin 2\omega_1 t' + \omega_2 x_2 \sin 2\omega_2 t' - \omega_1 x_1 \left[ \sin(\omega_1 + \omega_2) t' + \sin(\omega_2 - \omega_1) t' \right] \right. \\
 &\quad \left. - \omega_2 x_2 \left[ \sin(\omega_1 + \omega_2) t' + \sin(\omega_1 - \omega_2) t' \right] \right\} dt' \\
 &= \frac{1}{2} \epsilon_0 E_0^2 \left\{ -\frac{1}{2} x_1 \cos 2\omega_1 t' \Big|_0^t - \frac{1}{2} x_2 \cos 2\omega_2 t' \Big|_0^t + \frac{\omega_1 x_1 + \omega_2 x_2}{\omega_1 + \omega_2} \cos(\omega_1 + \omega_2) t' \Big|_0^t \right. \\
 &\quad \left. - \frac{\omega_2 x_2 - \omega_1 x_1}{\omega_2 - \omega_1} \cos(\omega_2 - \omega_1) t' \Big|_0^t \right\} \\
 &= \frac{1}{2} \epsilon_0 E_0^2 \left\{ -\frac{1}{2} x_1 \cos 2\omega_1 t + \frac{1}{2} x_1 - \frac{1}{2} x_2 \cos 2\omega_2 t + \frac{1}{2} x_2 + \frac{\omega_1 x_1 + \omega_2 x_2}{\omega_1 + \omega_2} \left[ \cos(\omega_1 + \omega_2) t - 1 \right] \right. \\
 &\quad \left. - \frac{\omega_2 x_2 - \omega_1 x_1}{\omega_2 - \omega_1} \left[ \cos(\omega_2 - \omega_1) t - 1 \right] \right\}
 \end{aligned}$$

Note that  $\cos 2\omega_1 t = \cos(2m-1)\Delta\omega t$  has an integer number of periods between  $t=0$  and  $t=T=2\pi/\Delta\omega$ ; therefore, time-averaging over the interval  $(0, T)$  eliminates the term containing  $\cos 2\omega_1 t$ . Similarly,  $\cos 2\omega_2 t = \cos(2m+1)\Delta\omega t$  averages out to zero. The same is true of  $\cos(\omega_1 + \omega_2)t = \cos 2m\Delta\omega t$  and  $\cos(\omega_1 - \omega_2)t = \cos \Delta\omega t$ . Consequently,

Dipoles' Energy Density, time-averaged over the interval  $(0, T) =$

$$\begin{aligned}
 &\frac{1}{2} \epsilon_0 E_0^2 \left( \frac{1}{2} x_1 + \frac{1}{2} x_2 - \frac{\omega_1 x_1 + \omega_2 x_2}{\omega_1 + \omega_2} + \frac{\omega_2 x_2 - \omega_1 x_1}{\omega_2 - \omega_1} \right) = \\
 &\frac{1}{2} \epsilon_0 E_0^2 \left\{ \frac{1}{2} x(\omega_1) + \frac{1}{2} x(\omega_2) - \frac{(m-1/2)\Delta\omega x(\omega_1) + (m+1/2)\Delta\omega x(\omega_2)}{2m\Delta\omega} + \frac{(m+1/2)\Delta\omega x(\omega_2) - (m-1/2)\Delta\omega x(\omega_1)}{\Delta\omega} \right\} \\
 &= \frac{1}{2} \epsilon_0 E_0^2 \left\{ \frac{x(\omega_1) - x(\omega_2)}{4m} + \frac{x(\omega_1) + x(\omega_2)}{2} + (m\Delta\omega) \frac{x(\omega_2) - x(\omega_1)}{\Delta\omega} \right\}
 \end{aligned}$$



Continuity equations at  $z = -d$ :

$$E\text{-field: } E_{oy}^{(a)} e^{-ik_0 \sigma_y^{(a)} d} + E_{oy}^{(b)} e^{-ik_0 \sigma_y^{(b)} d} = E_{oy}^{(t)} e^{-ik_0 \sigma_y^{(t)} d} \Rightarrow$$

$$a e^{-k_0 d \sqrt{n_0^2 \Lambda^2 \alpha - 1}} + b e^{+k_0 d \sqrt{n_0^2 \Lambda^2 \alpha - 1}} = \tau e^{ik_0 n_0 d \cos \theta}$$

$$H_x\text{-field: } \frac{i \sqrt{n_0^2 \Lambda^2 \alpha - 1}}{z_0} E_{oy}^{(a)} e^{-k_0 d \sqrt{n_0^2 \Lambda^2 \alpha - 1}} - \frac{i \sqrt{n_0^2 \Lambda^2 \alpha - 1}}{z_0} E_{oy}^{(b)} e^{+k_0 d \sqrt{n_0^2 \Lambda^2 \alpha - 1}}$$

$$= \frac{n_0 \cos \theta}{z_0} E_{oy}^{(t)} e^{ik_0 n_0 d \cos \theta}$$

$$\Rightarrow \frac{i \sqrt{n_0^2 \Lambda^2 \alpha - 1}}{n_0 \cos \theta} (a e^{-k_0 d \sqrt{n_0^2 \Lambda^2 \alpha - 1}} - b e^{+k_0 d \sqrt{n_0^2 \Lambda^2 \alpha - 1}}) = \tau e^{ik_0 n_0 d \cos \theta}$$

d) Let  $c = \frac{i \sqrt{n_0^2 \Lambda^2 \alpha - 1}}{n_0 \cos \theta}$ . Then the first two equations can be solved

for  $a$  and  $b$  in terms of  $r$ , as follows:

$$\begin{cases} a + b = 1 + r \\ a - b = \frac{1}{c}(1 - r) \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2}(1 + \frac{1}{c}) + \frac{r}{2}(1 - \frac{1}{c}) \\ b = \frac{1}{2}(1 - \frac{1}{c}) + \frac{r}{2}(1 + \frac{1}{c}) \end{cases}$$

The 3rd and 4th equations yield a relation between  $a$  and  $b$  when their left-hand-sides are set equal to each other:

$$a e^{-k_0 d \sqrt{\dots}} + b e^{+k_0 d \sqrt{\dots}} = c (a e^{-k_0 d \sqrt{\dots}} - b e^{+k_0 d \sqrt{\dots}}) \Rightarrow$$

$$a e^{-k_0 d \sqrt{\dots}} (c - 1) = b e^{+k_0 d \sqrt{\dots}} (c + 1) \Rightarrow$$

$$\left[ \frac{c+1}{2c} + \frac{r(c-1)}{2c} \right] e^{-2k_0 d \sqrt{n_0^2 \Lambda^2 \alpha - 1}} (c-1) = \left[ \frac{c-1}{2c} + \frac{r(c+1)}{2c} \right] (c+1) \Rightarrow$$

$$\left[ (c^2 - 1) + r(c-1)^2 \right] e^{-2k_0 d \sqrt{n_0^2 \Lambda^2 \alpha - 1}} = (c^2 - 1) + r(c+1)^2 \Rightarrow$$

$$(c^2 - 1)(1 - e^{-2k_0 d \sqrt{\dots}}) = r \left[ (c-1)^2 e^{-2k_0 d \sqrt{\dots}} - (c+1)^2 \right] \Rightarrow$$

