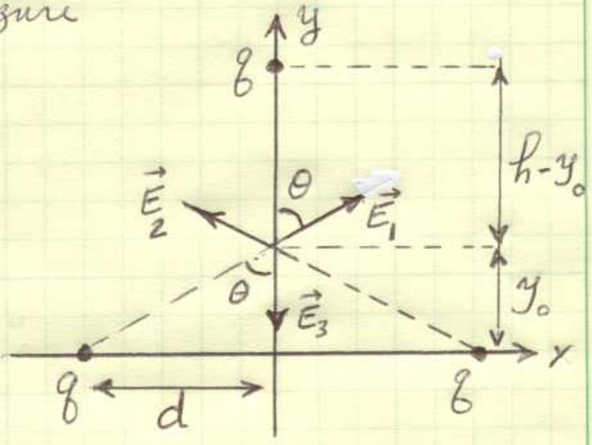
1) a) Because of symmetry, the point of equilibrium is likely to be on the y-axis, therefore, $x_0 = 0$. We also expect $0 < y_0 < h$, otherwise the point will be outside the triangle, with all the charges pushing or pulling in the same direction. From the figure it is readily seen that, if the three bisectors of the equilateral triangle are drawn, they'll meet at a central point which is equi-distant from the three charges. Therefore, $|\vec{E}_1| = |\vec{E}_2| = |\vec{E}_3|$.



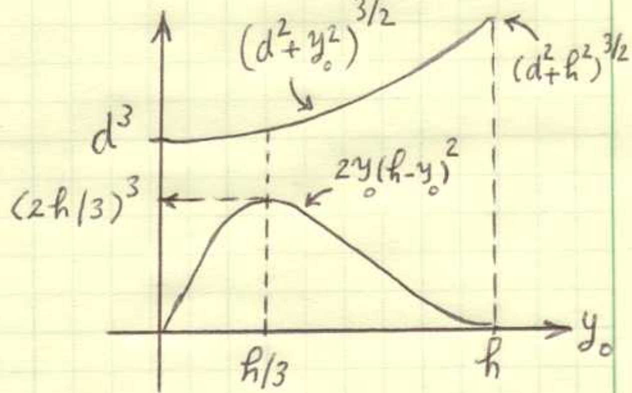
Also, $\theta = 60°$, which means that the three field vectors have an angular separation of $120°$ from each other. The net field, $\vec{E}_1 + \vec{E}_2 + \vec{E}_3$, is therefore equal to zero. We'll have:

$$\tan\theta = \frac{d}{y_0} \Rightarrow y_0 = \frac{d}{\tan 60°} = \frac{d}{\sqrt{3}} = \frac{\sqrt{3}\,d}{3} = h/3 \Rightarrow (x_0, y_0) = (0, \tfrac{h}{3})$$

*
Digression: A more formal solution recognizes that $|\vec{E}_1| = |\vec{E}_2|$ because of symmetry. Equilibrium then demands that $|\vec{E}_3| = 2|\vec{E}_1|\cos\theta \Rightarrow$

$$\frac{q}{(h-y_0)^2} = 2\frac{q}{d^2+y_0^2}\cdot\frac{y_0}{\sqrt{d^2+y_0^2}} \Rightarrow (d^2+y_0^2)^{3/2} = 2y_0(h-y_0)^2$$

The two functions appearing in the above equation are plotted on the right. Clearly, there are no crossing points unless



$(2h/3)^3 \geq (d^2 + \frac{h^2}{3^2})^{3/2} \Rightarrow h \geq \sqrt{3}\,d$.

In general, when $h > \sqrt{3}\,d$, there will be two solutions for y_0. For an equilateral triangle, however, $h=\sqrt{3}d$, and there is only one solution, namely, $y_0 = h/3$. (Note: When $h<\sqrt{3}d$, the above analysis does not imply that no equilibrium points exist; one must then re-examine the original assumption that $x_0=0$.)

b) There are no stable points of equilibrium in any electrostatic field. The reason is that, if a test charge placed at (x_0, y_0) is moved away from (x_0, y_0) by a small distance in an arbitrary direction, stability requires that the $\vec{E}$-field must always push it back toward (x_0, y_0). This means that the integral of $\vec{E}$ over a closed surface surrounding (x_0, y_0) must be either positive (i.e., all surrounding $\vec{E}$-fields pointing away from the equilibrium point) or negative (i.e., all surrounding $\vec{E}$-fields pointing toward the equilibrium point). In other words, $\vec{\nabla}\cdot\vec{E} \neq 0$ at the point of equilibrium. This is not possible, however, because, according to Gauss's law, $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$. Since there are no charges at the point (x_0, y_0), we have $\rho=0$ and, consequently, $\vec{\nabla}\cdot\vec{E}=0$, contradicting the assumption of stability.

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2) a) Surface area $= 4\pi R^2 \Rightarrow Q = 4\pi R^2 \sigma_0$

b) Velocity of sphere's surface at $\vec{r}'$: $\vec{V}(\vec{r}') = (R\sin\theta')\,\omega\,\hat{\phi}$.

$$\Rightarrow \vec{J}_s(\vec{r}') = \sigma_0 \vec{V}(\vec{r}') = R\sin\theta'\,\omega\,\sigma_0\,\hat{\phi}$$

c) $|\vec{r}-\vec{r}'|^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 = (0 - R\sin\theta'\cos\phi')^2 + (y - R\sin\theta'\sin\phi')^2 + (z - R\cos\theta')^2 = R^2\cancel{\sin^2\theta'}\cos^2\phi' + y^2 + R^2\cancel{\sin^2\theta'}\sin^2\phi' - 2Ry\sin\theta'\sin\phi' + z^2 + R^2\cancel{\cos^2\theta'} - 2Rz\cos\theta' = R^2 + y^2 + z^2 - 2Ry\sin\theta'\sin\phi' - 2Rz\cos\theta'$

$$\Rightarrow |\vec{r}-\vec{r}'| = \sqrt{(y^2+z^2+R^2) - 2Ry\sin\theta'\sin\phi' - 2Rz\cos\theta'}$$

$$1/|\vec{r}-\vec{r}'| = \left[(y^2+z^2+R^2)\left(1-2\,\frac{Ry\sin\theta'\sin\phi' + Rz\cos\theta'}{y^2+z^2+R^2}\right)\right]^{-1/2}$$

$$\simeq (y^2+z^2+R^2)^{-1/2}\left[1+\frac{Ry\sin\theta'\sin\phi' + Rz\cos\theta'}{y^2+z^2+R^2}\right]$$

$$\simeq \frac{1}{\sqrt{y^2+z^2}} + \frac{Ry}{(y^2+z^2)^{3/2}}\sin\theta'\sin\phi' + \frac{Rz}{(y^2+z^2)^{3/2}}\cos\theta'$$

d) $$A_\phi(\vec{r})\hat{\phi} = \frac{\mu_0}{4\pi}\int_{V'}\frac{\vec{J}(\vec{r}')\,dv'}{|\vec{r}-\vec{r}'|} = \frac{\mu_0}{4\pi}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\frac{J_s(\vec{r}')\sin\phi'\,\hat{\phi}}{|\vec{r}-\vec{r}'|}\,R^2\sin\theta'\,d\theta'\,d\phi'$$

$$\Rightarrow A_\phi(\vec{r}) = \frac{\mu_0}{4\pi}R^3\omega\sigma_0\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}|\vec{r}-\vec{r}'|^{-1}\sin\phi'\sin^2\theta'\,d\theta'\,d\phi'$$

e) Using $r=|\vec{r}|=\sqrt{y^2+z^2}$, $y=r\sin\theta$, $z=r\cos\theta$, we can write:

$1/|\vec{r}-\vec{r}'| \simeq \frac{1}{r} + \frac{R\sin\theta}{r^2}\sin\theta'\sin\phi' + \frac{R\cos\theta}{r^2}\cos\theta'$. Therefore,

$$A_\phi(\vec{r}) \cong \frac{\mu_0}{(4\pi)^2}(4\pi R^2\sigma_0)(R\omega)\left\{\frac{1}{r}\int_{\theta'=0}^{\pi}\sin^2\theta'\,d\theta'\cancelto{0}{\int_{\phi'=0}^{2\pi}\sin\phi'\,d\phi'} + \frac{R\sin\theta}{r^2}\int_{\theta'=0}^{\pi}\sin^3\theta'\,d\theta'\int_{\phi'=0}^{2\pi}\sin^2\phi'\,d\phi'\right.$$

$$\left. + \frac{R\cos\theta}{r^2}\int_{\theta'=0}^{\pi}\cos\theta'\sin^2\theta'\,d\theta'\cancelto{0}{\int_{\phi'=0}^{2\pi}\sin\phi'\,d\phi'}\right\}$$

$$\Rightarrow A_\phi(\vec{r}) \cong \frac{\mu_0}{12\pi}Q(R^2\omega)\frac{\sin\theta}{r^2} \Rightarrow \vec{A}(\vec{r}) \cong \frac{\mu_0}{4\pi r^2}\left(\frac{1}{3}QR^2\omega\right)(\hat{z}\times\hat{r});\quad r \gg R$$

f) Comparison with $\vec{A}(\vec{r})$ for small current loop reveals that $\vec{m} = \frac{1}{3}QR^2\omega\hat{z}$

g) mass-density (per unit surface area) $= \dfrac{M}{4\pi R^2}$

Density of linear momentum at $\vec{r}\,' = (M/4\pi R^2)\vec{V}(\vec{r}\,') = \left(\dfrac{M}{4\pi R}\right)\sin\theta'\,\omega\,\hat{\phi}$

Distance between $\vec{r}\,'$ and the z-axis $= R\sin\theta' \leftarrow$ (Direction is $\hat{\rho}$ of Cylindrical Coordinates.)

$$\vec{L} = \int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\left(\frac{M}{4\pi R}\right)\sin\theta'\,\omega(R\sin\theta')\hat{z}\;R^2\sin\theta'\,d\theta'\,d\phi' \quad \leftarrow (\hat{z} = \hat{\rho}\times\hat{\phi})$$

$$= \frac{MR^2\omega}{4\pi}\hat{z}\int_{\theta'=0}^{\pi}\sin^3\theta'\,d\theta'\int_{\phi'=0}^{2\pi}d\phi' \Rightarrow \vec{L} = \frac{2}{3}MR^2\omega\hat{z}$$

h) $\vec{m} = \frac{1}{3}QR^2\omega\hat{z} = \frac{1}{2}\left(\frac{Q}{M}\right)\vec{L}$

*Digression: For an electron, $e = -1.6\times10^{-19}$ C and $M = 9.11\times10^{-31}$ Kg

$\Rightarrow \vec{m} = -0.88\times10^{11}\,\vec{L} = -0.927\times10^{-23}\,(\vec{L}/\hbar) \leftarrow (\hbar = \frac{h}{2\pi}$ is planck's Constant$)$

The Constant $\mu_B = 0.927\times10^{-23}$ Amp$\cdot$m^2 is known as a Bohr magneton, the quantum of magnetic moment. In quantum mechanics, it turns out that the correct expression for $\vec{m}$ of an electron due to its spin angular momentum is nearly twice the value given by the above equation. In other words, the spin angular momentum of an electron is $\frac{1}{2}\hbar$, whereas its magnetic dipole moment is $\sim\mu_B$.

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3) a) $\epsilon(\omega) \cong 1 - \left(\frac{\omega_p}{\omega}\right)^2 > 0$ when $\omega > \omega_p$. $\Rightarrow n(\omega) = \sqrt{\epsilon(\omega)} =$ real and positive. Since the imaginary part of $n(\omega)$ at $\omega > \omega_p$ is zero, the material is transparent.

b) $V_p = \dfrac{c}{n(\omega)} = c/\sqrt{1-(\omega_p/\omega)^2}$. Since the denominator is less than unity, the phase velocity V_p is greater than c.

c) $n(\omega)=\sqrt{\epsilon(\omega)} = \left(1-\frac{\omega_p^2}{\omega^2+i\gamma\omega}\right)^{1/2}$

$$\frac{dn(\omega)}{d\omega} = \frac{1}{2}\,\frac{\omega_p^2(2\omega+i\gamma)}{(\omega^2+i\gamma\omega)^2}\left(1-\frac{\omega_p^2}{\omega^2+i\gamma\omega}\right)^{-1/2}$$

$$\Rightarrow n(\omega)+\omega\frac{dn(\omega)}{d\omega} = \left(1-\frac{\omega_p^2}{\omega^2+i\gamma\omega}\right)^{1/2} + \frac{\omega_p^2(\omega+\frac{i}{2}\gamma)}{\omega(\omega+i\gamma)^2}\left(1-\frac{\omega_p^2}{\omega^2+i\gamma\omega}\right)^{-1/2}$$

$$= \left(1-\frac{\omega_p^2}{\omega^2+i\gamma\omega}\right)^{-1/2}\left\{\frac{\omega^2-\omega_p^2+i\gamma\omega}{\omega^2+i\gamma\omega} + \frac{\omega_p^2(\omega+\frac{i}{2}\gamma)}{\omega(\omega+i\gamma)^2}\right\}$$

$$= \left(1-\frac{\omega_p^2}{\omega^2+i\gamma\omega}\right)^{-1/2}\left[1-\frac{1}{2}\left(\frac{\omega_p}{\omega}\right)^2\left(\frac{\gamma}{\omega}\right)\frac{\frac{\gamma}{\omega}+i}{1+(\gamma/\omega)^2}\right]$$

In the limit $\omega \gg \gamma$, we can omit the terms that contain (γ/ω) to obtain:

$$n(\omega)+\omega\frac{dn(\omega)}{d\omega} \simeq \left(1-\frac{\omega_p^2}{\omega^2}\right)^{-1/2}$$

Therefore, $V_g = \dfrac{c}{n(\omega)+\omega n'(\omega)} \simeq \sqrt{1-(\omega_p/\omega)^2}\, c < c$ (when $\omega > \omega_p$).

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4) a) Incident $\sigma_x=\sigma_y=0$, $\sigma_z=-1$. $\vec{E}_0 = E_x\hat{x}$, $Z_0\vec{H}_0 = \vec{\sigma}\times\vec{E}_0 = -E_x\hat{y}$

Reflected: $\sigma_x'=\sigma_y'=0$, $\sigma_z'=+1$; $\vec{E}_0' = rE_x\hat{x}$; $Z_0\vec{H}_0' = \vec{\sigma}'\times\vec{E}_0' = rE_x\hat{y}$

Transmitted: $\begin{cases} \sigma_x''=\sigma_y''=0;\ \vec{\sigma}''\cdot\vec{\sigma}'' = \epsilon(\omega) \Rightarrow \sigma_z''^2 = 1-(\omega_p/\omega)^2 \Rightarrow \sigma_z'' = -i\sqrt{(\omega_p/\omega)^2-1} \\ \vec{E}_0'' = \tau E_x\hat{x};\ Z_0\vec{H}_0'' = \vec{\sigma}''\times\vec{E}_0'' = -i\sqrt{(\omega_p/\omega)^2-1}\,\tau E_x\,\hat{y} \end{cases}$

Using the relevant $\vec{E}_0, \vec{H}_0, \vec{\sigma}$ for each beam (see above), the distributions will have the form $\vec{E}_0\exp[ik_0(\vec{\sigma}\cdot\vec{r}-ct)] = \vec{E}_0\exp(i\frac{\omega}{c}\sigma_z z)\exp(-i\omega t)$ and $\vec{H}_0\exp(i\frac{\omega}{c}\sigma_z z)\exp(-i\omega t)$.

b) Continuity of $\vec{E}_\parallel$: $E_x + rE_x = \tau E_x \Rightarrow 1+r=\tau$

Continuity of $\vec{H}_\parallel$: $-E_x + rE_x = -i\sqrt{(\omega_p/\omega)^2-1}\,\tau E_x \Rightarrow 1-r = i\sqrt{(\omega_p/\omega)^2-1}\,\tau$

$$\Rightarrow 1-r = i\sqrt{(\omega_p/\omega)^2-1}\,(1+r) \Rightarrow r = \frac{1-i\sqrt{(\omega_p/\omega)^2-1}}{1+i\sqrt{(\omega_p/\omega)^2-1}}$$

$$\tau = 1+r = \frac{2}{1+i\sqrt{(\omega_p/\omega)^2-1}}$$

c) $$R = \left|\frac{1-i\sqrt{(\omega_p/\omega)^2-1}}{1+i\sqrt{(\omega_p/\omega)^2-1}}\right|^2 = \frac{|1-i\sqrt{\cdots}|^2}{|1+i\sqrt{\cdots}|^2} = \frac{1+(\omega_p/\omega)^2-1}{1+(\omega_p/\omega)^2-1} = 1 \quad (\text{i.e., } 100\%)$$

d) $$\langle\vec{S}(\vec{r},t)\rangle = \frac{1}{2}\,\mathrm{Re}\,(\vec{E}''\times\vec{H}''^{*}) = \frac{1}{2}\,\mathrm{Re}\left\{\vec{E}_0''\, e^{k_0\sqrt{(\omega_p/\omega)^2-1}\,z} \times \vec{H}_0''^{*}\, e^{k_0\sqrt{(\omega_p/\omega)^2-1}\,z}\right\}$$

$$= \frac{\exp(2k_0\sqrt{(\omega_p/\omega)^2-1}\,z)}{2Z_0}\,\mathrm{Re}\left\{\tau E_x\hat{x}\times(+i\sqrt{(\omega_p/\omega)^2-1}\,\tau^* E_x^*\hat{y})\right\} \Rightarrow$$

$$\langle\vec{S}(\vec{r},t)\rangle = \frac{\exp(2k_0\sqrt{(\omega_p/\omega)^2-1}\,z)}{2Z_0}\,|\tau|^2|E_x|^2\sqrt{(\omega_p/\omega)^2-1}\;\overset{0}{\cancel{\mathrm{Re}(i)}}\,\hat{z} = 0 \checkmark$$

This is consistent with part (c), because 100% reflectance does not leave any energy to be transmitted across the interface.

e) The $\vec{E}$- and $\vec{H}$-fields in the plasma-like medium decay exponentially as $\exp(k_0\sqrt{(\omega_p/\omega)^2-1}\,z)$. The 1/e point of these fields occurs at:

$$\Delta z = \frac{1}{k_0\sqrt{(\omega_p/\omega)^2-1}} = \frac{\lambda_0}{2\pi\sqrt{(\omega_p/\omega)^2-1}}$$ ← skin depth (or penetration depth)

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5) a) Case of s-polarization,

$$r_s = \frac{\cos\theta-\sqrt{n^2-\sin^2\theta}}{\cos\theta+\sqrt{n^2-\sin^2\theta}} \xrightarrow[n\to\infty]{} \frac{\cos\theta-n}{\cos\theta+n} \longrightarrow \frac{-n}{+n} = -1 \checkmark$$

Incident: $\sigma_x = \sin\theta,\ \sigma_y = 0,\ \sigma_z = -\cos\theta;\ \vec{E}_s = E_y\hat{y};\ Z_0\vec{H}_0 = \vec{\sigma}\times\vec{E}_s = E_y(\cos\theta\hat{x}+\sin\theta\hat{z})$

Reflected: $\sigma_x' = \sin\theta,\ \sigma_y' = 0,\ \sigma_z' = +\cos\theta;\ \vec{E}_s' = r_s E_y\hat{y} = -E_y\hat{y};\ Z_0\vec{H}_0' = \vec{\sigma}'\times\vec{E}_s' = E_y(\cos\theta\hat{x}-\sin\theta\hat{z})$

Continuity of $\vec{E}_\parallel$ at the surface: $\vec{E}_s+\vec{E}_s' = E_y\hat{y}+r_s E_y\hat{y} = (E_y-E_y)\hat{y} = 0.$

The $\vec{E}_{\parallel}$-field is zero in the air just above the mirror, and also in the metal just beneath the surface. This is not inconsistent with the existence of a surface current, because the conductivity of the mirror is ∞.

Continuity of $\vec{H}_{\parallel}$ at the surface: $H_x\hat{x} + H'_x\hat{x} = \frac{E_s}{Z_0}(\cos\theta + \cos\theta)\hat{x}$ $= \frac{2E_s\cos\theta}{Z_0}\hat{x}$. The spatio-temporal dependence of the fields is $\exp(ik_0\vec{\sigma}\cdot\vec{r} - i\omega t)$. The reflected and incident beams have $\sigma_x = \sigma'_x = \sin\theta$ and, at the mirror surface, $z = 0$. The common spatio-temporal factor at the mirror surface is thus $\exp(ik_0\sigma_x x - i\omega t)$. The total magnetic field at the top of the mirror surface is $\frac{2E_s\cos\theta}{Z_0}\exp(ik_0\sin\theta\, x - i\omega t)\hat{x}$. The $\vec{H}$-field beneath the surface is zero (perfect conductor). The discontinuity of the H-field is then equal to the surface current density, that is,

$$\vec{J}_s(\vec{r}, t) = \frac{2E_s\cos\theta}{Z_0} e^{ik_0(\sin\theta\, x - ct)}\hat{y}$$

← use $\vec{\nabla}\times\vec{H} = \vec{J}$ to see that $\vec{J}_s$ is along $\hat{y}$.

There is no $\perp$ $\vec{E}$-field at the surface, therefore, $\sigma_s(\vec{r}, t) = 0$. This can also be inferred from $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$, because $\vec{\nabla}\cdot\vec{J}_s = 0$.

b) Case of p-polarization: $r_p = \frac{\sqrt{n^2 - \sin^2\theta} - n^2\cos\theta}{\sqrt{n^2 - \sin^2\theta} + n^2\cos\theta} \xrightarrow[n\to\infty]{} \frac{n - n^2\cos\theta}{n + n^2\cos\theta} \to \frac{-n^2\cos\theta}{n^2\cos\theta} = -1$ ✓

Incident: $\sigma_x = \sin\theta$, $\sigma_y = 0$, $\sigma_z = -\cos\theta$; $\vec{E}_p = E_p(\cos\theta\hat{x} + \sin\theta\hat{z})$, $Z_0\vec{H}_0 = \vec{\sigma}\times\vec{E}_p = -E_p\hat{y}$

Reflected: $\sigma'_x = \sin\theta$, $\sigma'_y = 0$, $\sigma'_z = +\cos\theta$; $\vec{E}'_p = -E_p(\cos\theta\hat{x} - \sin\theta\hat{z})$; $Z_0\vec{H}'_0 = \vec{\sigma}'\times\vec{E}'_p = -E_p\hat{y}$

$\vec{H}$-field at the top surface of mirror $= -\frac{2E_p}{Z_0}\exp(ik_0\sin\theta\, x - i\omega t)\hat{y} \Rightarrow$

$$\vec{J}_s(\vec{r}, t) = \frac{2E_p}{Z_0}\exp[ik_0(\sin\theta\, x - ct)]\,\hat{x}$$

← use $\vec{\nabla}\times\vec{H} = \vec{J}$ to see that $\vec{J}_s$ is along $\hat{x}$.

$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow \sigma_s(\vec{r}, t) = \epsilon_0(E_\perp^{\text{above}} - E_\perp^{\text{below}}) = \epsilon_0 E_z^{\text{above}} = 2\epsilon_0 E_p\sin\theta\exp[ik_0(\sin\theta\, x - ct)]$

It is readily verified that $\vec{\nabla}\cdot\vec{J}_s + \frac{\partial}{\partial t}\sigma_s = 0$.