Problem 1)

a) $\boldsymbol{p}(t) = -qx(t)\hat{\boldsymbol{x}}$. Therefore,

$$\frac{d\mathcal{E}(t)}{dt} = \boldsymbol{E}(t)\cdot\frac{d\boldsymbol{p}(t)}{dt} = q\omega|x_0|E_{x0}\cos(\omega t)\sin(\omega t - \varphi_0). \quad (1)$$

b) $$\frac{d\mathcal{E}_K(t)}{dt} = mv_x(t)\frac{dv_x(t)}{dt} = m\left[\frac{dx(t)}{dt}\right]\left[\frac{d^2x(t)}{dt^2}\right] = m\omega^3|x_0|^2\sin(\omega t-\varphi_0)\cos(\omega t-\varphi_0). \quad (2)$$

c) $$\frac{d\mathcal{E}_P(t)}{dt} = \alpha x(t)\frac{dx(t)}{dt} = -\alpha\omega|x_0|^2\cos(\omega t-\varphi_0)\sin(\omega t-\varphi_0). \quad (3)$$

d) $$\frac{d\mathcal{E}_L(t)}{dt} = \beta\left[\frac{dx(t)}{dt}\right]^2 = \beta\omega^2|x_0|^2\sin^2(\omega t-\varphi_0). \quad (4)$$

e) $$\frac{d\mathcal{E}_{\text{total}}(t)}{dt} = \frac{d}{dt}[\mathcal{E}_K(t)+\mathcal{E}_P(t)+\mathcal{E}_L(t)]$$

$$= \omega|x_0|^2\sin(\omega t-\varphi_0)\,[m\omega^2\cos(\omega t-\varphi_0) - \alpha\cos(\omega t-\varphi_0) + \beta\omega\sin(\omega t-\varphi_0)]$$

$$= m\omega|x_0|^2\sin(\omega t-\varphi_0)\,[(\omega^2-\omega_0^2)\cos(\omega t-\varphi_0) + \gamma\omega\sin(\omega t-\varphi_0)]. \quad (5)$$

Now, according to the Lorentz oscillator model, we have

$$|x_0|\exp(\mathrm{i}\varphi_0) = \frac{(q/m)E_{x0}}{\omega^2-\omega_0^2+\mathrm{i}\gamma\omega} = \left(\frac{qE_{x0}}{m}\right)\frac{(\omega^2-\omega_0^2)-\mathrm{i}\gamma\omega}{(\omega^2-\omega_0^2)^2+(\gamma\omega)^2}. \quad (6)$$

Consequently,

$$|x_0| = (qE_{x0}/m)/\sqrt{(\omega^2-\omega_0^2)^2+(\gamma\omega)^2}\,; \quad (7)$$

$$\cos(\varphi_0) = (\omega^2-\omega_0^2)/\sqrt{(\omega^2-\omega_0^2)^2+(\gamma\omega)^2}\,; \quad (8)$$

$$\sin(\varphi_0) = -\gamma\omega/\sqrt{(\omega^2-\omega_0^2)^2+(\gamma\omega)^2}. \quad (9)$$

Equation (5) may now be rewritten, as follows:

$$\frac{d\mathcal{E}_{\text{total}}(t)}{dt} = m\omega|x_0|^2\sin(\omega t-\varphi_0)$$

$$\times\sqrt{(\omega^2-\omega_0^2)^2+(\gamma\omega)^2}\,\underbrace{[\cos(\varphi_0)\cos(\omega t-\varphi_0)-\sin(\varphi_0)\sin(\omega t-\varphi_0)]}_{\cos a\cos b - \sin a\sin b = \cos(a+b)}$$

$$= q\omega|x_0|E_{x0}\cos(\omega t)\sin(\omega t-\varphi_0). \quad (10)$$

A comparison of Eq.(10) with Eq.(1) reveals that $d\mathcal{E}(t)/dt = d[\mathcal{E}_K(t)+\mathcal{E}_P(t)+\mathcal{E}_L(t)]/dt$.

Problem 2)

a) Denoting the vacuum wave-number of the incident and reflected plane-waves by $k_0 = \omega/c$, the incident $\boldsymbol{E}$ and $\boldsymbol{H}$ fields inside the glass prism are given by

$$\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r},t) = E_0[(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{z}})\pm\mathrm{i}\hat{\boldsymbol{y}}]\exp[\mathrm{i}(nk_0\sin\theta\,x - nk_0\cos\theta\,z-\omega t)]. \quad (1)$$

$$\boldsymbol{H}^{\mathrm{i}}(\boldsymbol{r},t) = \left(\frac{nE_0}{Z_0}\right)[-\hat{\boldsymbol{y}}\pm\mathrm{i}(\cos\theta\,\hat{\boldsymbol{x}}+\sin\theta\,\hat{\boldsymbol{z}})]\exp[\mathrm{i}(nk_0\sin\theta\,x-nk_0\cos\theta\,z-\omega t)]. \quad (2)$$

b) The reflected beam is similar to the incident beam, except that the signs of its k_z, E_z, H_x, and H_y are reversed, while its corresponding field amplitudes are multiplied by ρ_p and ρ_s, as follows:

$$\boldsymbol{E}^{\mathrm{r}}(\boldsymbol{r},t) = E_0\big[\rho_p(\cos\theta\,\hat{\boldsymbol{x}} - \sin\theta\,\hat{\boldsymbol{z}}) \pm \mathrm{i}\rho_s\hat{\boldsymbol{y}}\big]\exp[\mathrm{i}(nk_0\sin\theta\,x + nk_0\cos\theta\,z - \omega t)]. \quad (3)$$

$$\boldsymbol{H}^{\mathrm{r}}(\boldsymbol{r},t) = \left(\frac{nE_0}{Z_0}\right)\big[\rho_p\hat{\boldsymbol{y}} \mp \mathrm{i}\rho_s(\cos\theta\,\hat{\boldsymbol{x}} - \sin\theta\,\hat{\boldsymbol{z}})\big]\exp[\mathrm{i}(nk_0\sin\theta\,x + nk_0\cos\theta\,z - \omega t)]. \quad (4)$$

c) The dispersion relation in free space, namely, $k^2 = k_x^2 + k_z^2 = k_0^2 = (\omega/c)^2$, yields the following value for the transmitted (evanescent) beam's k_z:

$$k_z^{\mathrm{t}} = -\sqrt{k_0^2 - k_x^2} = -\sqrt{k_0^2 - (nk_0\sin\theta)^2} = -\mathrm{i}k_0\sqrt{(n\sin\theta)^2 - 1}. \quad (5)$$

The components of the transmitted E-field are then found to be

$$E_{x0}^{\mathrm{t}} = \tau_p E_{x0}^{\mathrm{i}} = \tau_p E_0\cos\theta, \quad (6a)$$

$$E_{y0}^{\mathrm{t}} = \tau_s E_{y0}^{\mathrm{i}} = \pm\mathrm{i}\tau_s E_0, \quad (6b)$$

$$E_{z0}^{t} = -\frac{k_x E_{x0}^{\mathrm{t}}}{k_z^{\mathrm{t}}} = \frac{n\tau_p E_0\sin\theta\cos\theta}{\mathrm{i}\sqrt{(n\sin\theta)^2-1}}. \quad (6c)$$

$$\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t) = E_0\left[\tau_p\cos\theta\left(\hat{\boldsymbol{x}} - \frac{\mathrm{i}n\sin\theta}{\sqrt{n^2\sin^2\theta-1}}\hat{\boldsymbol{z}}\right) \pm \mathrm{i}\tau_s\hat{\boldsymbol{y}}\right]$$
$$\times\exp\left(k_0\sqrt{n^2\sin^2\theta - 1}\,z\right)\exp[\mathrm{i}(nk_0\sin\theta\,x - \omega t)]. \quad (7)$$

d) $$\boldsymbol{\nabla}\times\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t) = -\,\partial\boldsymbol{B}^{\mathrm{t}}(\boldsymbol{r},t)/\partial t \quad \rightarrow \quad \boldsymbol{k}^{\mathrm{t}}\times\boldsymbol{E}_0^{\mathrm{t}} = \omega\mu_0\boldsymbol{H}_0^{\mathrm{t}}. \quad (8)$$

$$\mu_0\omega\boldsymbol{H}_0^{\mathrm{t}} = \left(nk_0\sin\theta\,\hat{\boldsymbol{x}} - \mathrm{i}k_0\sqrt{n^2\sin^2\theta - 1}\,\hat{\boldsymbol{z}}\right)\times E_0\left[\tau_p\cos\theta\,\hat{\boldsymbol{x}} \pm \mathrm{i}\tau_s\hat{\boldsymbol{y}} - \frac{\mathrm{i}n\tau_p\sin\theta\cos\theta}{\sqrt{n^2\sin^2\theta-1}}\hat{\boldsymbol{z}}\right]. \quad (9)$$

$$\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r},t) = \left(\frac{E_0}{Z_0}\right)\left[\pm\tau_s\left(-\sqrt{n^2\sin^2\theta - 1}\,\hat{\boldsymbol{x}} + \mathrm{i}n\sin\theta\,\hat{\boldsymbol{z}}\right) + \tau_p\left(\frac{\mathrm{i}\cos\theta}{\sqrt{n^2\sin^2\theta-1}}\right)\hat{\boldsymbol{y}}\right]$$
$$\times\exp\left(k_0\sqrt{n^2\sin^2\theta - 1}\,z\right)\exp[\mathrm{i}(nk_0\sin\theta\,x - \omega t)]. \quad (10)$$

e) $$\langle\boldsymbol{S}^{\mathrm{t}}(\boldsymbol{r},t)\rangle = \tfrac{1}{2}\mathrm{Re}[\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r},t)\times\boldsymbol{H}^{\mathrm{t}*}(\boldsymbol{r},t)]$$
$$= \left(\frac{|E_0|^2}{2Z_0}\right)\mathrm{Re}\left[n\sin\theta\left(|\tau_s|^2 + \frac{|\tau_p|^2\cos^2\theta}{n^2\sin^2\theta-1}\right)\hat{\boldsymbol{x}} \pm \mathrm{i}n\tau_p\tau_s^*\sin(2\theta)\,\hat{\boldsymbol{y}}\right.$$
$$\left. + \mathrm{i}\sqrt{n^2\sin^2\theta - 1}\left(|\tau_s|^2 - \frac{|\tau_p|^2\cos^2\theta}{n^2\sin^2\theta-1}\right)\hat{\boldsymbol{z}}\right]\exp\left(2k_0\sqrt{n^2\sin^2\theta - 1}\,z\right). \quad (11)$$

It is now clear that $\langle S_x\rangle$, a real-valued positive entity, does not depend on the sense of circular polarization. In contrast, $\langle S_y\rangle$ switches sign depending on the sense of circular polarization, that is, whether the plus or minus sign is used to describe the incident beam. The time-averaged z-component of the Poynting vector, $\langle S_z\rangle$, vanishes because the corresponding entity inside the brackets in Eq.(11) is purely imaginary. All three components of the Poynting vector decay exponentially with the distance z from the bottom of the prism, as dictated by the exponential factor appearing on the right-hand-side of Eq.(11).

Digression: The non-vanishing of the $\langle S_x\rangle$ of the evanescent field is true for both *p*- and *s*-polarized incident light, as it is true for circularly- and elliptically-polarized beams. It indicates the existence of a certain (small but measurable) *forward* shift of the reflected beam's footprint relative to that of the incident beam, which is known as the Goos-Hänchen effect. In contrast, the non-vanishing of $\langle S_y \rangle$ occurs exclusively for circularly- or elliptically-polarized incident light; it indicates the existence of a small but measurable *lateral* shift of the footprint of the reflected beam relative to that of the incident beam. This lateral shift, whose direction depends on the sense of the circular (or elliptical) incident polarization, is known as the Imbert-Fedorov effect.

Problem 3)

a) $$\boldsymbol{k} = k_\rho \hat{\boldsymbol{\rho}} + k_z \hat{\boldsymbol{z}} = (\omega/c)\sin(\theta_0)\,\hat{\boldsymbol{\rho}} + (\omega/c)\cos(\theta_0)\,\hat{\boldsymbol{z}}. \tag{1}$$

b) $$E_\rho(\boldsymbol{r},t) = -E_0\cos\theta_0\cos(\varphi-\varphi_0)\exp\{\mathrm{i}[k_\rho\rho\cos(\varphi-\varphi_0) + k_z z - \omega t]\}, \tag{2a}$$

$$E_\varphi(\boldsymbol{r},t) = E_0\cos\theta_0\sin(\varphi-\varphi_0)\exp\{\mathrm{i}[k_\rho\rho\cos(\varphi-\varphi_0) + k_z z - \omega t]\}, \tag{2b}$$

$$E_z(\boldsymbol{r},t) = E_0\sin\theta_0\exp\{\mathrm{i}[k_\rho\rho\cos(\varphi-\varphi_0) + k_z z - \omega t]\}. \tag{2c}$$

$$H_\rho(\boldsymbol{r},t) = -(E_0/Z_0)\sin(\varphi-\varphi_0)\exp\{\mathrm{i}[k_\rho\rho\cos(\varphi-\varphi_0) + k_z z - \omega t]\}, \tag{3a}$$

$$H_\varphi(\boldsymbol{r},t) = -(E_0/Z_0)\cos(\varphi-\varphi_0)\exp\{\mathrm{i}[k_\rho\rho\cos(\varphi-\varphi_0) + k_z z - \omega t]\}, \tag{3b}$$

$$H_z(\boldsymbol{r},t) = 0. \tag{3c}$$

c) The following integrals will be useful in subsequent derivations:

i) $$\int_0^{2\pi}\sin\varphi\exp(\mathrm{i}x\cos\varphi)\,d\varphi = -\left(\frac{1}{\mathrm{i}x}\right)\exp(\mathrm{i}x\cos\varphi)\Big|_{\varphi=0}^{2\pi} = -\frac{\exp(\mathrm{i}x)-\exp(\mathrm{i}x)}{\mathrm{i}x} = 0. \tag{4}$$

ii) $$\frac{d}{dx}\int_0^{2\pi}\exp(\mathrm{i}x\cos\varphi)\,d\varphi = \int_0^{2\pi}\mathrm{i}\cos\varphi\exp(\mathrm{i}x\cos\varphi)\,d\varphi = 2\pi J_0'(x) = -2\pi J_1(x)$$

$$\rightarrow \quad \int_0^{2\pi}\cos\varphi\exp(\mathrm{i}x\cos\varphi)\,d\varphi = \mathrm{i}2\pi J_1(x). \tag{5}$$

To find the (E_ρ, E_φ, E_z) and (H_ρ, H_φ, H_z) of the superposition, we integrate the fields obtained in part (b) over φ_0 from 0 to 2π. We find

$$E_\rho^{(\text{total})} = \int_{\varphi_0=0}^{2\pi} E_\rho(\boldsymbol{r},t)d\varphi_0 = -E_0\cos\theta_0\exp[\mathrm{i}(k_z z-\omega t)]\int_0^{2\pi}\cos\varphi\exp(\mathrm{i}k_\rho\rho\cos\varphi)\,d\varphi$$
$$= -\mathrm{i}2\pi E_0\cos\theta_0 J_1(k_\rho\rho)\exp[\mathrm{i}(k_z z-\omega t)], \tag{6a}$$

$$E_\varphi^{(\text{total})} = \int_{\varphi_0=0}^{2\pi} E_\varphi(\boldsymbol{r},t)d\varphi_0 = -E_0\cos\theta_0\exp[\mathrm{i}(k_z z-\omega t)]\int_0^{2\pi}\sin\varphi\exp(\mathrm{i}k_\rho\rho\cos\varphi)\,d\varphi = 0, \tag{6b}$$

$$E_z^{(\text{total})} = \int_{\varphi_0=0}^{2\pi} E_z(\boldsymbol{r},t)d\varphi_0 = E_0\sin\theta_0\exp[\mathrm{i}(k_z z-\omega t)]\int_0^{2\pi}\exp(\mathrm{i}k_\rho\rho\cos\varphi)\,d\varphi$$
$$= 2\pi E_0\sin\theta_0 J_0(k_\rho\rho)\exp[\mathrm{i}(k_z z-\omega t)]. \tag{6c}$$

$$H_\rho^{(\text{total})} = \int_{\varphi_0=0}^{2\pi} H_\rho(\boldsymbol{r},t)d\varphi_0 = (E_0/Z_0)\exp[\mathrm{i}(k_z z-\omega t)]\int_0^{2\pi}\sin\varphi\exp(\mathrm{i}k_\rho\rho\cos\varphi)\,d\varphi = 0, \tag{7a}$$

$$H_\varphi^{(\text{total})} = \int_{\varphi_0=0}^{2\pi} H_\varphi(\boldsymbol{r},t)d\varphi_0 = -(E_0/Z_0)\exp[\mathrm{i}(k_z z-\omega t)]\int_0^{2\pi}\cos\varphi\exp(\mathrm{i}k_\rho\rho\cos\varphi)\,d\varphi$$
$$= -\mathrm{i}2\pi(E_0/Z_0)J_1(k_\rho\rho)\exp[\mathrm{i}(k_z z-\omega t)], \tag{7b}$$

$$H_z^{(\text{total})} = \int_{\varphi_0=0}^{2\pi} H_z(\boldsymbol{r},t)d\varphi_0 = 0. \tag{7c}$$
