Problem 1)

a) 1) $\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\text{free}}$,

2) $\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\text{free}}+\cancelto{0}{\frac{\partial\boldsymbol{D}}{\partial t}}$,

3) $\boldsymbol{\nabla}\times\boldsymbol{E}=-\cancelto{0}{\frac{\partial\boldsymbol{B}}{\partial t}}$,

4) $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$

b) The first and third equations thus form the set of equations for electrostatics, namely,

$$\varepsilon_{\text{o}}\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r})=\rho_{\text{free}}(\boldsymbol{r})-\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r}),$$
$$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r})=0.$$

Similarly, the second and fourth equations form the set of equations for magnetostatics, that is,

$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r})=\boldsymbol{J}_{\text{free}}(\boldsymbol{r}),$$
$$\mu_{\text{o}}\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r})=-\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r}).$$

c) The sources of the electrostatic field $\boldsymbol{E}(\boldsymbol{r})$ are the free and bound electric charge densities $\rho_{\text{free}}(\boldsymbol{r})$ and $-\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r})$, respectively.

d) The sources of the magnetostatic field $\boldsymbol{H}$ are the free electric current-density $\boldsymbol{J}_{\text{free}}$ and the bound magnetic charge density $-\boldsymbol{\nabla}\cdot\boldsymbol{M}$.

Problem 2)

a) 1) $\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_{\text{free}}$,

2) $\boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_{\text{free}}+\frac{\partial\boldsymbol{D}}{\partial t}$,

3) $\boldsymbol{\nabla}\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}\quad\rightarrow\quad\boldsymbol{\nabla}\times(\varepsilon_{\text{o}}\boldsymbol{E}+\boldsymbol{P})=-\varepsilon_{\text{o}}\frac{\partial(\mu_{\text{o}}\boldsymbol{H}+\boldsymbol{M})}{\partial t}+\boldsymbol{\nabla}\times\boldsymbol{P}$

$\rightarrow\quad\boldsymbol{\nabla}\times\boldsymbol{D}=-\varepsilon_{\text{o}}\left(\frac{\partial\boldsymbol{M}}{\partial t}-\varepsilon_{\text{o}}^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}\right)-\varepsilon_{\text{o}}\mu_{\text{o}}\frac{\partial\boldsymbol{H}}{\partial t}$

$\rightarrow\quad\boldsymbol{\nabla}\times\boldsymbol{D}=-\varepsilon_{\text{o}}\boldsymbol{J}_{\text{bound}}^{(\text{m})}-\varepsilon_{\text{o}}\mu_{\text{o}}\frac{\partial\boldsymbol{H}}{\partial t}$,

4) $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0\quad\rightarrow\quad\mu_{\text{o}}\boldsymbol{\nabla}\cdot\boldsymbol{H}=-\boldsymbol{\nabla}\cdot\boldsymbol{M}\quad\rightarrow\quad\mu_{\text{o}}\boldsymbol{\nabla}\cdot\boldsymbol{H}=\rho_{\text{bound}}^{(\text{m})}$.

b) In the above equations, $\rho_{\text{bound}}^{(\text{m})}=-\boldsymbol{\nabla}\cdot\boldsymbol{M}$ and $\boldsymbol{J}_{\text{bound}}^{(\text{m})}=\partial\boldsymbol{M}/\partial t-\varepsilon_{\text{o}}^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}$. We may thus write

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{bound}}^{(\text{m})}=\boldsymbol{\nabla}\cdot\left(\frac{\partial\boldsymbol{M}}{\partial t}-\varepsilon_{\text{o}}^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}\right)=\frac{\partial(\boldsymbol{\nabla}\cdot\boldsymbol{M})}{\partial t}-\varepsilon_{\text{o}}^{-1}\cancelto{0}{\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{P})}=\frac{\partial(\boldsymbol{\nabla}\cdot\boldsymbol{M})}{\partial t}=-\frac{\partial\rho_{\text{bound}}^{(\text{m})}}{\partial t}$$

$$\rightarrow\quad\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{bound}}^{(\text{m})}+\frac{\partial\rho_{\text{bound}}^{(\text{m})}}{\partial t}=0.$$

Problem 3) The incident beam is circularly polarized, which means that its *p*- and *s*-components are equal in magnitude and 90° apart in phase. Since $\theta = 45°$, $\sin\theta = \cos\theta = 1/\sqrt{2}$.

a) $$\rho_p = \frac{\sqrt{n^2-\sin^2\theta}-n^2\cos\theta}{\sqrt{n^2-\sin^2\theta}+n^2\cos\theta} = \frac{\sqrt{1.5^2-½}-1.5^2/\sqrt{2}}{\sqrt{1.5^2-½}+1.5^2/\sqrt{2}} \cong \frac{1.323-1.591}{1.323+1.591} \cong -0.092,$$

$$\rho_s = \frac{\cos\theta-\sqrt{n^2-\sin^2\theta}}{\cos\theta+\sqrt{n^2-\sin^2\theta}} = \frac{(1/\sqrt{2})-\sqrt{1.5^2-½}}{(1/\sqrt{2})+\sqrt{1.5^2-½}} \cong \frac{0.707-1.323}{0.707+1.323} \cong -0.303.$$

The reflectivity of the dielectric surface for the incident (circularly-polarized) beam is thus given by $R = ½\left(|\rho_p|^2 + |\rho_s|^2\right) \cong 0.05.$

b) The polarization state of the reflected beam is elliptical, because the reflected *p*- and *s*-components continue to have a 90° phase difference (i.e., same as the incident beam), but the two amplitudes are no longer equal: $E_s^{(r)}/E_p^{(r)} = 0.303/0.092 \cong 3.29$. The sense of rotation of the *E*-field around the ellipse is the reverse of that of the incident beam, so that a right-circularly-polarized incident beam will result in a left-elliptically-polarized reflected beam, and vice-versa.

c) The Fresnel transmission coefficients are readily computed as follows:

$$\tau_p = \frac{2\sqrt{n^2-\sin^2\theta}}{\sqrt{n^2-\sin^2\theta}+n^2\cos\theta} = \frac{2\sqrt{1.5^2-½}}{\sqrt{1.5^2-½}+1.5^2/\sqrt{2}} \cong \frac{2\times 1.323}{1.323+1.591} \cong 0.908,$$

$$\tau_s = \frac{2\cos\theta}{\cos\theta+\sqrt{n^2-\sin^2\theta}} = \frac{\sqrt{2}}{(1/\sqrt{2})+\sqrt{1.5^2-½}} \cong \frac{1.414}{0.707+1.323} \cong 0.697.$$

d) Since $\tau_s = E_{y0}^{(t)}/E_{y0}^{(i)} = E_s^{(t)}/E_s^{(i)}$, the *s*-component of the transmitted *E*-field amplitude is $\tau_s = 0.697$ times the *s*-component of the incident *E*-field amplitude. However, with the *p*-component we have $\tau_p = E_{x0}^{(t)}/E_{x0}^{(i)} = (E_p^{(t)}\cos\theta')/(E_p^{(i)}\cos\theta)$. Using Snell's law, $\sin\theta = n\sin\theta'$, we find $\theta' = 28.126°$. Therefore, $\cos\theta\,/\cos\theta' \cong 0.802$, and $E_p^{(t)}/E_p^{(i)} \cong 0.728$. As was the case with the reflected beam, we see that the transmitted *p*- and *s*-components have unequal magnitudes: $E_s^{(t)}/E_p^{(t)} \cong 0.697/0.728 \cong 0.957$. The phase difference between $E_s^{(t)}$ and $E_p^{(t)}$ is still 90°, which is the phase difference between the *s*- and *p*-components of the incident wave. We conclude that the transmitted beam is elliptically polarized, albeit not too far from circular, having the same sense of rotation of the *E*-field around the ellipse as that of the incident beam.

e) The magnitude of the time-averaged Poynting vector is $\langle S\rangle = ½n|E_0|^2/Z_0$, where Z_0 is the impedance of free space. This means that the Poynting vector of the transmitted *p*-light is $1.5\times 0.728^2 = 0.795$ times that of the incident beam. Similarly, the Poynting vector of the transmitted *s*-light is $1.5\times 0.697^2 = 0.729$ times that of the incident beam. The total Poynting vector (i.e., p plus s) of the transmitted beam is, therefore, $½(0.795+0.729) \cong 0.762$ times that of the incident beam.

f) In contrast to the reflected beam, which has the same cross-sectional area as the incident beam, the cross-sectional area of the transmitted beam is greater than that of the incident beam by $\cos\theta'/\cos\theta \cong 1.247$. The transmitted optical power is, therefore, $0.762\times 1.247 \cong 0.95$ times the incident optical power. In part (a) we found the reflected optical power to be 0.05 times the incident power. Conservation of energy is thus confirmed.

Problem 4) a) Denoting the magnitude of the k-vector in free space by $k_0 = \omega/c$, we have

$$\boldsymbol{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}} = (k_0 \sin\theta \cos\phi)\hat{\boldsymbol{x}} + (k_0 \sin\theta \sin\phi)\hat{\boldsymbol{y}} + (k_0 \cos\theta)\hat{\boldsymbol{z}},$$

$$\boldsymbol{E}_0 = E_{x0}\hat{\boldsymbol{x}} + E_{y0}\hat{\boldsymbol{y}} + E_{z0}\hat{\boldsymbol{z}},$$

$$\boldsymbol{H}_0 = H_{x0}\hat{\boldsymbol{x}} + H_{y0}\hat{\boldsymbol{y}} + H_{z0}\hat{\boldsymbol{z}}.$$

b) $$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0 \quad\rightarrow\quad \boldsymbol{k}\cdot\boldsymbol{E}_0 = 0 \quad\rightarrow\quad k_xE_{x0} + k_yE_{y0} + k_zE_{z0} = 0$$

$$\rightarrow\quad E_{z0} = -\big(k_xE_{x0} + k_yE_{y0}\big)/k_z.$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t \quad\rightarrow\quad i\boldsymbol{k}\times\boldsymbol{E}_0 = i\omega\mu_0\boldsymbol{H}_0$$

$$\rightarrow\quad \boldsymbol{H}_0 = (\mu_0\omega)^{-1}\boldsymbol{k}\times\boldsymbol{E}_0 = (\mu_0\omega)^{-1}\big(k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}}\big)\times\big(E_{x0}\hat{\boldsymbol{x}} + E_{y0}\hat{\boldsymbol{y}} + E_{z0}\hat{\boldsymbol{z}}\big)$$

$$\rightarrow\quad H_{x0} = \big(k_yE_{z0} - k_zE_{y0}\big)/(\mu_0\omega),$$

$$H_{y0} = \big(k_zE_{x0} - k_xE_{z0}\big)/(\mu_0\omega),$$

$$H_{z0} = \big(k_xE_{y0} - k_yE_{x0}\big)/(\mu_0\omega).$$

The field components $E_{z0}, H_{x0}, H_{y0}, H_{z0}$ are thus determined once the components E_{x0} and E_{y0} are specified.

c) $$\boldsymbol{S}(\boldsymbol{r},t) = \mathrm{Re}\,\{\boldsymbol{E}_0\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\}\times\mathrm{Re}\,\{\boldsymbol{H}_0\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\}$$

$$= [\boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) - \boldsymbol{E}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\times[\boldsymbol{H}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) - \boldsymbol{H}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$$

$$= (\boldsymbol{E}_0'\times\boldsymbol{H}_0')\cos^2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) + (\boldsymbol{E}_0''\times\boldsymbol{H}_0'')\sin^2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$
$$-(\boldsymbol{E}_0'\times\boldsymbol{H}_0'' + \boldsymbol{E}_0''\times\boldsymbol{H}_0')\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$$

$$= ½(\boldsymbol{E}_0'\times\boldsymbol{H}_0' + \boldsymbol{E}_0''\times\boldsymbol{H}_0'') + ½(\boldsymbol{E}_0'\times\boldsymbol{H}_0' - \boldsymbol{E}_0''\times\boldsymbol{H}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$$
$$-½(\boldsymbol{E}_0'\times\boldsymbol{H}_0'' + \boldsymbol{E}_0''\times\boldsymbol{H}_0')\sin[2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)].$$

Noting that $\boldsymbol{k}$ is a real-valued vector, we will have

$$\boldsymbol{E}_0'\times\boldsymbol{H}_0' \pm \boldsymbol{E}_0''\times\boldsymbol{H}_0'' = (\mu_0\omega)^{-1}[\boldsymbol{E}_0'\times(\boldsymbol{k}\times\boldsymbol{E}_0') \pm \boldsymbol{E}_0''\times(\boldsymbol{k}\times\boldsymbol{E}_0'')]$$
$$= (\mu_0\omega)^{-1}[(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')\boldsymbol{k} - \cancelto{0}{(\boldsymbol{k}\cdot\boldsymbol{E}_0')}\boldsymbol{E}_0' \pm (\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\boldsymbol{k} \mp \cancelto{0}{(\boldsymbol{k}\cdot\boldsymbol{E}_0'')}\boldsymbol{E}_0'']$$
$$= (\mu_0\omega)^{-1}(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' \pm \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\boldsymbol{k}$$

$$\boldsymbol{E}_0'\times\boldsymbol{H}_0'' + \boldsymbol{E}_0''\times\boldsymbol{H}_0' = (\mu_0\omega)^{-1}[\boldsymbol{E}_0'\times(\boldsymbol{k}\times\boldsymbol{E}_0'') + \boldsymbol{E}_0''\times(\boldsymbol{k}\times\boldsymbol{E}_0')]$$
$$= 2(\mu_0\omega)^{-1}(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'')\boldsymbol{k}$$

Therefore,

$$\boldsymbol{S}(\boldsymbol{r},t) = ½(\mu_0\omega)^{-1}\{(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'') + (\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' - \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$$
$$-2(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'')\sin[2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\}\boldsymbol{k}.$$

d) The electromagnetic momentum density in free space is given by $\boldsymbol{p}(\boldsymbol{r},t) = \boldsymbol{S}(\boldsymbol{r},t)/c^2$.

e) Assuming that $\boldsymbol{E}_0$ is real-valued, we may write the energy-density of the E-field as follows:

$$\begin{aligned}
\mathcal{E}_{\mathrm{E}}(\boldsymbol{r},t) &= ½\varepsilon_0 \mathrm{Re}\{\boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\cdot \mathrm{Re}\{\boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\\
&= ½\varepsilon_0 \boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\cdot \boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\\
&= ½\varepsilon_0 \boldsymbol{E}_0'\cdot \boldsymbol{E}_0'\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\\
&= ¼\varepsilon_0 \boldsymbol{E}_0'\cdot \boldsymbol{E}_0' + ¼\varepsilon_0 \boldsymbol{E}_0'\cdot \boldsymbol{E}_0'\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].
\end{aligned}$$

Similarly, the energy-density of the H-field is given by

$$\begin{aligned}
\mathcal{E}_{\mathrm{H}}(\boldsymbol{r},t) &= ½\mu_0 \mathrm{Re}\{\boldsymbol{H}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\cdot \mathrm{Re}\{\boldsymbol{H}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\\
&= ¼\mu_0 \boldsymbol{H}_0'\cdot \boldsymbol{H}_0' + ¼\mu_0 \boldsymbol{H}_0'\cdot \boldsymbol{H}_0'\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].
\end{aligned}$$

The above expression for the H-field energy-density may be further simplified by noting that

$$\begin{aligned}
\boldsymbol{H}_0'\cdot\boldsymbol{H}_0' &= (\mu_0\omega)^{-2}(\boldsymbol{k}\times\boldsymbol{E}_0')\cdot(\boldsymbol{k}\times\boldsymbol{E}_0') = (\mu_0\omega)^{-2}[(\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0') - (\cancelto{0}{\boldsymbol{k}\cdot\boldsymbol{E}_0'})^2]\\
&= (\mu_0\omega)^{-2}(\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0') = (\varepsilon_0/\mu_0)\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'.
\end{aligned}$$

It is seen that $\mathcal{E}_{\mathrm{H}}(\boldsymbol{r},t) = \mathcal{E}_{\mathrm{E}}(\boldsymbol{r},t)$. The total energy-density of the fields is, therefore, given by

$$\mathcal{E}(\boldsymbol{r},t) = \mathcal{E}_{\mathrm{E}}(\boldsymbol{r},t) + \mathcal{E}_{\mathrm{H}}(\boldsymbol{r},t) = ½\varepsilon_0\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + ½\varepsilon_0\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].$$

f) Poynting's theorem asserts that, in free space, $\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{r},t) + \partial\mathcal{E}(\boldsymbol{r},t)/\partial t = 0$. Recalling that $\boldsymbol{E}_0'' = 0$, the results obtained in parts (c) and (e) above now yield

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{r},t) &= \boldsymbol{\nabla}\cdot\{½(\mu_0\omega)^{-1}(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')\{1+\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\boldsymbol{k}\}\\
&= ½(\mu_0\omega)^{-1}(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')\boldsymbol{\nabla}\cdot\Big\{\{1+\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}(k_x\hat{\boldsymbol{x}}+k_y\hat{\boldsymbol{y}}+k_z\hat{\boldsymbol{z}})\Big\}\\
&= -(\mu_0\omega)^{-1}(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')(k_x^2+k_y^2+k_z^2)\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\\
&= -\varepsilon_0\omega\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].
\end{aligned}$$

$$\begin{aligned}
\partial\mathcal{E}(\boldsymbol{r},t)/\partial t &= \partial\{½\varepsilon_0\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + ½\varepsilon_0\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}/\partial t\\
&= \varepsilon_0\omega\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].
\end{aligned}$$

The energy continuity equation (i.e., Poynting's theorem) is thus seen to be satisfied.

Digression: This problem can be solved in the general case when $\boldsymbol{E}_0'' \neq 0$, although the algebra is a bit tedious. Below we derive the energy densities of the E- and H-fields in the general case.

$$\begin{aligned}
\mathcal{E}_{\mathrm{E}}(\boldsymbol{r},t) &= ½\varepsilon_0 \mathrm{Re}\,\{\boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\cdot \mathrm{Re}\,\{\boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\\
&= ½\varepsilon_0[\boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{E}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\cdot[\boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{E}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\\
&= ½\varepsilon_0[(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) + (\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\sin^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\\
&\qquad -2(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'')\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\\
&= ¼\varepsilon_0\{(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'') + (\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' - \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\\
&\qquad -2\boldsymbol{E}_0'\cdot\boldsymbol{E}_0''\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}.
\end{aligned}$$

Similarly,

$$\mathcal{E}_{\mathrm{H}}(\boldsymbol{r},t) = \tfrac{1}{2}\mu_0 \mathrm{Re}\{\boldsymbol{H}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\cdot \mathrm{Re}\{\boldsymbol{H}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}$$
$$= \tfrac{1}{4}\mu_0\{(\boldsymbol{H}_0'\cdot\boldsymbol{H}_0' + \boldsymbol{H}_0''\cdot\boldsymbol{H}_0'') + (\boldsymbol{H}_0'\cdot\boldsymbol{H}_0' - \boldsymbol{H}_0''\cdot\boldsymbol{H}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$$
$$-2\boldsymbol{H}_0'\cdot\boldsymbol{H}_0''\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}.$$

Now

$$\boldsymbol{H}_0'\cdot\boldsymbol{H}_0' \pm \boldsymbol{H}_0''\cdot\boldsymbol{H}_0'' = (\mu_0\omega)^{-2}[(\boldsymbol{k}\times\boldsymbol{E}_0')\cdot(\boldsymbol{k}\times\boldsymbol{E}_0') \pm (\boldsymbol{k}\times\boldsymbol{E}_0'')\cdot(\boldsymbol{k}\times\boldsymbol{E}_0'')]$$
$$= (\mu_0\omega)^{-2}[(\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0') - \cancelto{0}{(\boldsymbol{k}\cdot\boldsymbol{E}_0')}^2 \pm (\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'') \mp \cancelto{0}{(\boldsymbol{k}\cdot\boldsymbol{E}_0'')}^2]$$
$$= (\mu_0\omega)^{-2}(\boldsymbol{k}\cdot\boldsymbol{k})[(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0') \pm (\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')]$$
$$= (\varepsilon_0/\mu_0)[(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0') \pm (\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')]$$

$$\boldsymbol{H}_0'\cdot\boldsymbol{H}_0'' = (\mu_0\omega)^{-2}(\boldsymbol{k}\times\boldsymbol{E}_0')\cdot(\boldsymbol{k}\times\boldsymbol{E}_0'')$$
$$= (\mu_0\omega)^{-2}[(\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'') - \cancelto{0}{(\boldsymbol{k}\cdot\boldsymbol{E}_0')}\cancelto{0}{(\boldsymbol{k}\cdot\boldsymbol{E}_0'')}]$$
$$= (\mu_0\omega)^{-2}(\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'')$$
$$= (\varepsilon_0/\mu_0)(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'').$$

As before, the energy densities of the *E*- and *H*-fields are seen to be equal. We will have

$$\mathcal{E}_{\mathrm{E}}(\boldsymbol{r},t) + \mathcal{E}_{\mathrm{H}}(\boldsymbol{r},t) = \tfrac{1}{2}\varepsilon_0\{(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'') + (\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' - \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$$
$$-2\boldsymbol{E}_0'\cdot\boldsymbol{E}_0''\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}.$$

The Poynting vector was already derived in part (c) for the general case of $\boldsymbol{E}_0'' \neq 0$. Verification of the energy continuity equation now follows the same steps as in part (f).