**Problem 1**) a) An optical medium is linear when it responds linearly to the local electric and magnetic fields. For instance, in the case of monochromatic excitation with frequency $\omega$, a linear material's polarization and magnetization at $(\boldsymbol{r},t)$ are given by $\boldsymbol{P}(\boldsymbol{r})e^{-\mathrm{i}\omega t} = \varepsilon_0\chi_{\mathrm{e}}(\omega)\boldsymbol{E}(\boldsymbol{r})e^{-\mathrm{i}\omega t}$ and $\boldsymbol{M}(\boldsymbol{r})e^{-\mathrm{i}\omega t} = \mu_0\chi_{\mathrm{m}}(\omega)\boldsymbol{H}(\boldsymbol{r})e^{-\mathrm{i}\omega t}$, while $\rho_{\text{free}}(\boldsymbol{r},t) = 0$ and $\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t) = 0$. It is seen that the relation between $\boldsymbol{P}(\boldsymbol{r})$ and $\boldsymbol{E}(\boldsymbol{r})$ is one of proportionality, and so is the relation between $\boldsymbol{M}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$.

The material is said to be isotropic when its response to the local $\boldsymbol{E}$ and $\boldsymbol{H}$ fields does not depend on the direction of these fields. For instance, the aforementioned linear medium is also isotropic if $\chi_{\mathrm{e}}(\omega)$ as well as $\chi_{\mathrm{m}}(\omega)$ remain the same irrespective of whether the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields happen to be along the $x$, or $y$, or $z$ directions or, for that matter, along any direction in space.

The material medium is homogeneous if its optical properties, such as electric and magnetic susceptibilities $\chi_{\mathrm{e}}(\omega)$ and $\chi_{\mathrm{m}}(\omega)$, are independent of the location $\boldsymbol{r}$ within the medium.

b) The plane of incidence is the geometric plane defined by the vector $\boldsymbol{k}^{(\text{inc})}$ and the surface normal, which, in the present problem, is the $z$-axis. In the case of normal incidence, where the incident $k$-vector is aligned with the $z$-axis, the plane of incidence is not unique. Stated differently, in the case of normal incidence, any plane that is perpendicular to the interfacial $xy$-plane can be considered to be the plane of incidence.

c) The incident plane-wave is $p$-polarized ($p$ stands for parallel) if its state of polarization is linear, with the $E$-field vector residing entirely in the plane of incidence. The plane-wave is $s$-polarized ($s$ stands for senkrecht, which is German for perpendicular) if its state of polarization is also linear, but the $E$-field is perpendicular to the plane of incidence. At normal incidence, one cannot distinguish between $p$- and $s$-polarization states; consequently, when a normally-incident plane-wave is linearly polarized, it is equally valid to treat it as either $p$- or $s$-polarized light.

d) The incident plane-wave is linearly polarized if $E_p = 0$, or $E_s = 0$, or when neither $E_p$ nor $E_s$ is zero but $\varphi_p - \varphi_s = 0°$ or $180°$. The plane-wave is circularly polarized if $|E_p| = |E_s| \neq 0$ and $\varphi_p - \varphi_s = 90°$ or $-90°$. When a monochromatic plane-wave is neither linearly nor circularly polarized, it is said to be elliptically polarized. With the passage of time, the tip of the $E$-field vector in the latter case describes an ellipse, which is known as the ellipse of polarization.

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**Problem 2**) a)

$$\boldsymbol{E}(\boldsymbol{r},t) = \text{Real}\{\boldsymbol{E}_0 e^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\} = \text{Real}\{(\boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0'')e^{\mathrm{i}[(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')\cdot\boldsymbol{r}-\omega t]}\}$$

$$= e^{-\boldsymbol{k}''\cdot\boldsymbol{r}}\,\text{Real}\{(\boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0'')[\cos(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega t) + \mathrm{i}\sin(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega t)]\}$$

$$= e^{-\boldsymbol{k}''\cdot\boldsymbol{r}}\,[\boldsymbol{E}_0'\cos(\omega t - \boldsymbol{k}'\cdot\boldsymbol{r}) + \boldsymbol{E}_0''\sin(\omega t - \boldsymbol{k}'\cdot\boldsymbol{r})].$$

b) Attenuation of the $E$-field is due to the leading exponential factor $\exp(-\boldsymbol{k}''\cdot\boldsymbol{r})$ in the above equation. This attenuation, which is in the direction of the vector $\boldsymbol{k}''$, occurs at a rate given by the magnitude $k''$ of $\boldsymbol{k}''$. Whatever the magnitude of the $E$-field may be at a point $\boldsymbol{r}_0$ within a plane perpendicular to $\boldsymbol{k}''$, if one moves away from $\boldsymbol{r}_0$ by the distance of $1/k''$ along the unit-vector $\hat{\boldsymbol{k}}'' = \boldsymbol{k}''/k''$ (that is, if one moves to the point $\boldsymbol{r}_0 + (\boldsymbol{k}''/k''^2)$), then the $E$-field magnitude will drop by a factor of $1/e$. This is the sense in which the rate of attenuation of the $E$-field along the direction of $\hat{\boldsymbol{k}}''$ is said to be equal to the magnitude $k''$ of the vector $\boldsymbol{k}''$.

c) The phase is obtained by examining the argument $(\omega t - \boldsymbol{k}' \cdot \boldsymbol{r})$ of the sine and cosine functions in the expressions of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields. In any plane that is perpendicular to the vector $\boldsymbol{k}'$, the phase $\boldsymbol{k}' \cdot \boldsymbol{r}$ of a plane-wave is a constant. The phase difference between the fields in two planes that are both perpendicular to $\boldsymbol{k}'$ at a separation distance of $d$ (in the direction of $\boldsymbol{k}'$) thus equals $k'd$. If we consider a point $(\boldsymbol{r}_0, t_0)$ on a given phase-front and try to keep $(\omega t - \boldsymbol{k}' \cdot \boldsymbol{r})$ constant as $t$ rises from $t_0$ to $t_0 + \Delta t$, we must move from $\boldsymbol{r}_0$ to $\boldsymbol{r}_0 + \Delta \boldsymbol{r}$ in the direction of $\boldsymbol{k}'$ and in such a way as to ensure that $\omega \Delta t - \boldsymbol{k}' \cdot \Delta \boldsymbol{r} = 0$. The phase-front velocity is thus seen to be along the direction of $\boldsymbol{k}'$ and have the magnitude $v_{\text{phase}} = \Delta r / \Delta t = \omega / k'$.

d) The state of polarization of the plane-wave is determined by the two vectors $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$. If either one of these vectors happens to be zero, or if $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ turn out to be parallel or anti-parallel to each other (i.e., aligned along a straight line), the $E$-field will have a fix and unique direction in space at all times, in which case the plane-wave is said to be linearly-polarized along that direction. In contrast, when $\boldsymbol{E}_0'$ and $\boldsymbol{E}_0''$ are equal in magnitude and perpendicular to each other, the plane-wave is said to be circularly-polarized.

e) A plane-wave is homogeneous when its $k$-vector is real; that is, when $\boldsymbol{k}'' = 0$. Whereas the amplitudes of the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of an inhomogeneous plane-wave exponentially decay along the direction of $\boldsymbol{k}''$, in the case of homogeneous plane-waves, the field amplitudes only vary periodically in space along the direction of $\boldsymbol{k}'$—while also oscillating with frequency $\omega$ in time. The $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of a homogeneous plane-wave do *not* grow or decay in spacetime.

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**Problem 3)** a) $\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$, $\quad \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{H}_0 \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$.

b) i) $\boldsymbol{\nabla} \cdot \boldsymbol{D}(\boldsymbol{r}, t) = \rho_{\text{free}}(\boldsymbol{r}, t) \quad \rightarrow \quad \mathrm{i}\boldsymbol{k} \cdot \varepsilon_0 \varepsilon(\omega) \boldsymbol{E}_0 \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] = 0 \quad \rightarrow \quad \boldsymbol{k} \cdot \boldsymbol{E}_0 = 0.$

ii) $\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{J}_{\text{free}}(\boldsymbol{r}, t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r}, t)}{\partial t}$

$$\rightarrow \mathrm{i}\boldsymbol{k} \times \boldsymbol{H}_0 \cancel{e^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}} = -\mathrm{i}\omega \varepsilon_0 \varepsilon(\omega) \boldsymbol{E}_0 \cancel{e^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}} \quad \rightarrow \quad \boldsymbol{k} \times \boldsymbol{H}_0 = -\varepsilon_0 \varepsilon(\omega) \omega \boldsymbol{E}_0.$$

iii) $\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r}, t)}{\partial t}$

$$\rightarrow \mathrm{i}\boldsymbol{k} \times \boldsymbol{E}_0 \cancel{e^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}} = \mathrm{i}\omega \mu_0 \mu(\omega) \boldsymbol{H}_0 \cancel{e^{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}} \quad \rightarrow \quad \boldsymbol{k} \times \boldsymbol{E}_0 = \mu_0 \mu(\omega) \omega \boldsymbol{H}_0.$$

iv) $\boldsymbol{\nabla} \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0 \quad \rightarrow \quad \mathrm{i}\boldsymbol{k} \cdot \mu_0 \mu(\omega) \boldsymbol{H}_0 \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] = 0 \quad \rightarrow \quad \boldsymbol{k} \cdot \boldsymbol{H}_0 = 0.$

c) $\boldsymbol{k} \cdot \boldsymbol{E}_0 = 0 \quad \rightarrow \quad k_x E_{0x} + k_y E_{0y} + k_z E_{0z} = 0 \quad \rightarrow \quad E_{0z} = -(k_x E_{0x} + k_y E_{0y}) / k_z.$

d) Multiply both sides of Eq.(ii) into $\mu_0 \mu(\omega) \omega$, then substitute $\boldsymbol{k} \times \boldsymbol{E}_0$ for $\mu_0 \mu(\omega) \omega \boldsymbol{H}_0$ from Eq.(iii) to arrive at

from Eq.(i)

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_0) = -\mu_0 \varepsilon_0 \omega^2 \mu(\omega) \varepsilon(\omega) \boldsymbol{E}_0 \quad \rightarrow \quad \cancelto{0}{(\boldsymbol{k} \cdot \boldsymbol{E}_0)}\boldsymbol{k} - (\boldsymbol{k} \cdot \boldsymbol{k}) \cancel{\boldsymbol{E}_0} = -(\omega/c)^2 \mu(\omega) \varepsilon(\omega) \cancel{\boldsymbol{E}_0}$$

$$\rightarrow \quad \boldsymbol{k} \cdot \boldsymbol{k} = (\omega/c)^2 \mu(\omega) \varepsilon(\omega).$$

Traditionally, the refractive index is defined as $n(\omega) = \sqrt{\mu(\omega)\varepsilon(\omega)}$. Consequently, the above dispersion relation may equivalently be written as $k^2 = [n(\omega)\omega/c]^2$.

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**Problem 4**) a) The correct expression for the Poynting vector is $\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}'(\boldsymbol{r},t) \times \boldsymbol{H}'(\boldsymbol{r},t)$.

b)
$$\begin{aligned}\text{Real}\{\boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t)\} &= \text{Real}\{[\boldsymbol{E}'(\boldsymbol{r},t) + \text{i}\boldsymbol{E}''(\boldsymbol{r},t)] \times [\boldsymbol{H}'(\boldsymbol{r},t) + \text{i}\boldsymbol{H}''(\boldsymbol{r},t)]\}\\ &= \text{Real}\{[\boldsymbol{E}' \times \boldsymbol{H}' - \boldsymbol{E}'' \times \boldsymbol{H}''] + \text{i}[\boldsymbol{E}' \times \boldsymbol{H}'' + \boldsymbol{E}'' \times \boldsymbol{H}']\}\\ &= \boldsymbol{E}' \times \boldsymbol{H}' - \boldsymbol{E}'' \times \boldsymbol{H}''.\end{aligned}$$

The presence of $\boldsymbol{E}'' \times \boldsymbol{H}''$ causes the above expression to differ from that obtained in part (a), which indicates that the Poynting vector $\boldsymbol{S}(\boldsymbol{r},t)$ should *not* be written as $\text{Real}\{\boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t)\}$.

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**Problem 5**) a) In the dispersion relation, $\boldsymbol{k}\cdot\boldsymbol{k} = [n(\omega)\omega/c]^2$, we have $\boldsymbol{k}\cdot\boldsymbol{k} = (\boldsymbol{k}' + \text{i}\boldsymbol{k}'')\cdot(\boldsymbol{k}' + \text{i}\boldsymbol{k}'') = k'^2 - k''^2 + \text{i}2\boldsymbol{k}'\cdot\boldsymbol{k}''$. Since the expression on the right-hand side of the dispersion relation is real, equating it to $\boldsymbol{k}\cdot\boldsymbol{k}$ implies that the imaginary part $2\boldsymbol{k}'\cdot\boldsymbol{k}''$ of $\boldsymbol{k}\cdot\boldsymbol{k}$ must be zero; that is, $\boldsymbol{k}'$ is orthogonal to $\boldsymbol{k}''$. Equality of the real parts then yields $k'^2 - k''^2 = [n(\omega)\omega/c]^2$. Since the right-hand side of this equation is real and positive, we must have $k' > k''$.

b) $\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t \quad \rightarrow \quad \text{i}\boldsymbol{k} \times \boldsymbol{E}_0 \exp[\text{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)] = \text{i}\omega\mu_0\mu(\omega)\boldsymbol{H}_0 \exp[\text{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$

$\rightarrow \quad \boldsymbol{H}_0 = \dfrac{\boldsymbol{k}\times\boldsymbol{E}_0}{\mu_0\mu(\omega)\omega}$.

c)
$$\begin{aligned}\langle \boldsymbol{S}(\boldsymbol{r},t)\rangle &= ½\text{Real}\{\boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}^*(\boldsymbol{r},t)\}\\ &= ½\text{Real}\left\{\boldsymbol{E}_0 e^{\text{i}[(\boldsymbol{k}'+\text{i}\boldsymbol{k}'')\cdot\boldsymbol{r} - \omega t]} \times \boldsymbol{H}_0^* e^{-\text{i}[(\boldsymbol{k}'-\text{i}\boldsymbol{k}'')\cdot\boldsymbol{r} - \omega t]}\right\}\\ &= \frac{e^{-2\boldsymbol{k}''\cdot\boldsymbol{r}}}{2\mu_0\mu(\omega)\omega}\text{Real}\{\boldsymbol{E}_0 \times (\boldsymbol{k}^* \times \boldsymbol{E}_0^*)\} = \frac{e^{-2\boldsymbol{k}''\cdot\boldsymbol{r}}}{2\mu_0\mu(\omega)\omega}\text{Real}\{(\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*)\boldsymbol{k}^* - (\boldsymbol{E}_0\cdot\boldsymbol{k}^*)\boldsymbol{E}_0^*\}.\end{aligned}$$

Considering that $\boldsymbol{E}_0\cdot\boldsymbol{E}_0^* = (\boldsymbol{E}_0' + \text{i}\boldsymbol{E}_0'')\cdot(\boldsymbol{E}_0' - \text{i}\boldsymbol{E}_0'') = \boldsymbol{E}_0'^2 + \boldsymbol{E}_0''^2$ is real, the first term on the right-hand side of the above equation simplifies to $\text{Real}\{(\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*)\boldsymbol{k}^*\} = (\boldsymbol{E}_0'^2 + \boldsymbol{E}_0''^2)\boldsymbol{k}'$. As for the second term, we write

$$\begin{aligned}&\boldsymbol{k}^* \times (\boldsymbol{E}_0 \times \boldsymbol{E}_0^*) = (\boldsymbol{k}^*\cdot\boldsymbol{E}_0^*)\boldsymbol{E}_0 - (\boldsymbol{k}^*\cdot\boldsymbol{E}_0)\boldsymbol{E}_0^*\\ \rightarrow\ &(\boldsymbol{k}' - \text{i}\boldsymbol{k}'') \times [(\boldsymbol{E}_0' + \text{i}\boldsymbol{E}_0'') \times (\boldsymbol{E}_0' - \text{i}\boldsymbol{E}_0'')] = (\boldsymbol{k}\cdot\boldsymbol{E}_0)^*\boldsymbol{E}_0 - (\boldsymbol{E}_0\cdot\boldsymbol{k}^*)\boldsymbol{E}_0^*\\ \rightarrow\ &(\boldsymbol{k}' - \text{i}\boldsymbol{k}'') \times [-2\text{i}(\boldsymbol{E}_0' \times \boldsymbol{E}_0'')] = -(\boldsymbol{E}_0\cdot\boldsymbol{k}^*)\boldsymbol{E}_0^*\\ \rightarrow\ &2\boldsymbol{k}'' \times (\boldsymbol{E}_0' \times \boldsymbol{E}_0'') + \text{i}2\boldsymbol{k}' \times (\boldsymbol{E}_0' \times \boldsymbol{E}_0'') = (\boldsymbol{E}_0\cdot\boldsymbol{k}^*)\boldsymbol{E}_0^*.\end{aligned}$$

0 — In consequence of Maxwell's 1st equation

Thus, we find that $\text{Real}\{(\boldsymbol{E}_0\cdot\boldsymbol{k}^*)\boldsymbol{E}_0^*\} = 2\boldsymbol{k}'' \times (\boldsymbol{E}_0' \times \boldsymbol{E}_0'')$. Consequently,

$$\langle \boldsymbol{S}(\boldsymbol{r},t)\rangle = \frac{[(\boldsymbol{E}_0'^2 + \boldsymbol{E}_0''^2)\boldsymbol{k}' - 2\boldsymbol{k}''\times(\boldsymbol{E}_0' \times \boldsymbol{E}_0'')]\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})}{2\mu_0\mu(\omega)\omega}.$$

The first term of the above expression is along the direction of $\boldsymbol{k}'$, which, in accordance with the results obtained in part (a), is orthogonal to $\boldsymbol{k}''$. The second term of the expression is also seen to be orthogonal to $\boldsymbol{k}''$ (due to cross-multiplication). Therefore, the time-averaged Poynting vector has no component along the direction of $\boldsymbol{k}''$.

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