Problem 1) a) From Maxwell's first equation we have $\boldsymbol{k}\cdot\boldsymbol{E}_0 = k_z E_{zo} = 0$. Since $k_z \neq 0$, we must have $E_{zo} = E'_{zo} + \mathrm{i}E''_{zo} = 0$, which indicates that both E'_{zo} and E''_{zo} are equal to zero.

b) $$\boldsymbol{E}'_0 + \mathrm{i}\boldsymbol{E}''_0 = (E'_{xo}\hat{\boldsymbol{x}} + E'_{yo}\hat{\boldsymbol{y}}) + \mathrm{i}(E''_{xo}\hat{\boldsymbol{x}} + E''_{yo}\hat{\boldsymbol{y}}) = \underbrace{(E'_{xo} + \mathrm{i}E''_{xo})}_{E_{xo}}\hat{\boldsymbol{x}} + \underbrace{(E'_{yo} + \mathrm{i}E''_{yo})}_{E_{yo}}\hat{\boldsymbol{y}}$$

$$= \underbrace{\left(E'^2_{xo} + E''^2_{xo}\right)^{1/2}}_{|E_{xo}|}\exp[\mathrm{i}\underbrace{\tan^{-1}(E''_{xo}/E'_{xo})}_{\varphi_{xo}}]\,\hat{\boldsymbol{x}} + \underbrace{\left(E'^2_{yo} + E''^2_{yo}\right)^{1/2}}_{|E_{yo}|}\exp[\mathrm{i}\underbrace{\tan^{-1}(E''_{yo}/E'_{yo})}_{\varphi_{yo}}]\,\hat{\boldsymbol{y}}. \quad (1)$$

c) $$\varphi_{xo} - \varphi_{yo} = 0 \text{ or } \pm 180° \quad\rightarrow\quad \tan(\varphi_{xo} - \varphi_{yo}) = 0 \quad\rightarrow\quad \frac{\tan(\varphi_{xo}) - \tan(\varphi_{yo})}{1 + \tan(\varphi_{xo})\tan(\varphi_{yo})} = 0$$

$$\rightarrow\quad \tan(\varphi_{xo}) = \tan(\varphi_{yo}) \quad\rightarrow\quad E''_{xo}/E'_{xo} = E''_{yo}/E'_{yo} = \alpha. \leftarrow \boxed{\alpha \text{ is some real constant}} \quad (2)$$

We thus have $\boldsymbol{E}'_0 = E'_{xo}\hat{\boldsymbol{x}} + E'_{yo}\hat{\boldsymbol{y}}$ and $\boldsymbol{E}''_0 = E''_{xo}\hat{\boldsymbol{x}} + E''_{yo}\hat{\boldsymbol{y}} = \alpha(E'_{xo}\hat{\boldsymbol{x}} + E'_{yo}\hat{\boldsymbol{y}})$. This shows that $\boldsymbol{E}'_0$ and $\boldsymbol{E}''_0$ are parallel to each other when $\alpha > 0$, and are anti-parallel when $\alpha < 0$.

d) $$\varphi_{xo} - \varphi_{yo} = \pm 90° \quad\rightarrow\quad \tan(\varphi_{xo} - \varphi_{yo}) = \infty \quad\rightarrow\quad \frac{\tan(\varphi_{xo}) - \tan(\varphi_{yo})}{1 + \tan(\varphi_{xo})\tan(\varphi_{yo})} = \infty$$

$$\rightarrow\quad \tan(\varphi_{xo})\tan(\varphi_{yo}) = -1 \quad\rightarrow\quad E''_{xo}/E'_{xo} = -E'_{yo}/E''_{yo} = \beta. \leftarrow \boxed{\beta \text{ is some real constant}} \quad (3)$$

Since the magnitudes of E_{xo} and E_{yo} are also equal to each other, we have

$$|E_{xo}| = |E_{yo}| \;\rightarrow\; E'^2_{xo} + E''^2_{xo} = E'^2_{yo} + E''^2_{yo} \;\rightarrow\; E'^2_{xo}[1 + (E''_{xo}/E'_{xo})^2] = E''^2_{yo}[(E'_{yo}/E''_{yo})^2 + 1]$$

$$\rightarrow\quad E'^2_{xo}(1 + \beta^2) = E''^2_{yo}(\beta^2 + 1) \quad\rightarrow\quad E'_{xo} = \pm E''_{yo}. \quad (4)$$

From Eqs.(3) and (4), we now find that $E''_{xo} = \mp E'_{yo}$. Consequently,

$$|\boldsymbol{E}'_0|^2 = E'^2_{xo} + E'^2_{yo} = E''^2_{xo} + E''^2_{yo} = |\boldsymbol{E}''_0|^2; \quad (5)$$

$$\boldsymbol{E}'_0\cdot\boldsymbol{E}''_0 = E'_{xo}E''_{xo} + E'_{yo}E''_{yo} = 0. \quad (6)$$

The above equations confirm that the real-valued vectors $\boldsymbol{E}'_0$ and $\boldsymbol{E}''_0$ have equal magnitudes and are orthogonal to each other.

Problem 2) a) From Maxwell's 2nd equation with $\boldsymbol{J}_{\text{free}} = 0$ we have $\boldsymbol{\nabla}\times\boldsymbol{H} = \partial\boldsymbol{D}/\partial t$, which yields

$$[\mu_0\mu(\omega)]^{-1}\,\boldsymbol{\nabla}\times\boldsymbol{B} = \varepsilon_0\varepsilon(\omega)\,\partial\boldsymbol{E}/\partial t$$

$$\rightarrow\quad \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \mu_0\varepsilon_0\mu(\omega)\varepsilon(\omega)\,\partial(-\boldsymbol{\nabla}\psi - \partial\boldsymbol{A}/\partial t)/\partial t$$

$$\rightarrow\quad \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \boldsymbol{\nabla}^2\boldsymbol{A} = -[n^2(\omega)/c^2][\boldsymbol{\nabla}(\partial\psi/\partial t) + \partial^2\boldsymbol{A}/\partial t^2]$$

$$\rightarrow\quad \boldsymbol{\nabla}^2\boldsymbol{A} - [n(\omega)/c]^2\,\partial^2\boldsymbol{A}/\partial t^2 = \boldsymbol{\nabla}\{\boldsymbol{\nabla}\cdot\boldsymbol{A} + [n(\omega)/c]^2\,\partial\psi/\partial t\}. \quad (1)$$

In the Lorenz gauge, we set $\boldsymbol{\nabla}\cdot\boldsymbol{A} + [n(\omega)/c]^2\,\partial\psi/\partial t = 0$ to arrive at the following wave equation for the vector potential:

$$\boldsymbol{\nabla}^2\boldsymbol{A} - [n(\omega)/c]^2\,\partial^2\boldsymbol{A}/\partial t^2 = 0. \quad (2)$$

From Maxwell's first equation (with $\rho_{\text{free}} = 0$), working again in the Lorenz gauge, we find

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = 0 \quad\rightarrow\quad \varepsilon_0\varepsilon(\omega)\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0 \quad\rightarrow\quad \boldsymbol{\nabla}\cdot(-\boldsymbol{\nabla}\psi - \partial\boldsymbol{A}/\partial t) = 0$$

$$\rightarrow\quad \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\psi) + \partial(\boldsymbol{\nabla}\cdot\boldsymbol{A})/\partial t = 0 \quad\rightarrow\quad \nabla^2\psi - [n(\omega)/c]^2\,\partial^2\psi/\partial t^2 = 0. \tag{3}$$

b) The assumption of monochromaticity implies that the fields have a time-dependence factor $\exp(-\mathrm{i}\omega t)$. Consequently $\partial^2\boldsymbol{A}(\boldsymbol{r},t)/\partial t^2 = (-\mathrm{i}\omega)^2\boldsymbol{A}(\boldsymbol{r}) = -\omega^2\boldsymbol{A}(\boldsymbol{r})$; similarly, $\partial^2\psi(\boldsymbol{r},t)/\partial t^2 = -\omega^2\psi(\boldsymbol{r})$. For plane-wave solutions of Maxwell's equations, we now write $\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}_0\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$ and $\psi(\boldsymbol{r}) = \psi_0\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$. We will then have

$$\boldsymbol{\nabla}^2\boldsymbol{A}(\boldsymbol{r}) = -(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{A}_0\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}), \tag{4}$$

$$\nabla^2\psi(\boldsymbol{r}) = -(\boldsymbol{k}\cdot\boldsymbol{k})\psi_0\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}). \tag{5}$$

The wave equations for $\boldsymbol{A}(\boldsymbol{r},t)$ and $\psi(\boldsymbol{r},t)$ in Eqs.(2) and (3) now yield $\boldsymbol{k}\cdot\boldsymbol{k} = [n(\omega)/c]^2\omega^2$, which is the same dispersion relation $k^2 = [\omega n(\omega)/c]^2$ as obtained previously from Maxwell's equations *without* resort to the potentials.

c) The field amplitudes for $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$ and $\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{B}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$ may now be derived from $\boldsymbol{E} = -\boldsymbol{\nabla}\psi - \partial\boldsymbol{A}/\partial t$ as $\boldsymbol{E}_0 = -\mathrm{i}\boldsymbol{k}\psi_0 + \mathrm{i}\omega\boldsymbol{A}_0$, and from $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ as $\boldsymbol{B}_0 = \mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_0$.

Problem 3) Considering that $E_B = \rho_m E_A\exp(\mathrm{i}k_0 d)$, we will have

$$E_A = \tau_m E_0 + \rho_m E_B\exp(\mathrm{i}k_0 d) = \tau_m E_0 + \rho_m^2 E_A\exp(\mathrm{i}2k_0 d) \;\rightarrow\; E_A = \frac{\tau_m E_0}{1 - \rho_m^2\exp(\mathrm{i}2k_0 d)}. \tag{1}$$

The overall transmission coefficient τ of the Fabry-Perot cavity may now be determined straightforwardly, as follows:

$$\tau E_0 = \tau_m E_A\exp(\mathrm{i}k_0 d) \quad\rightarrow\quad \tau = \frac{\tau_m^2\exp(\mathrm{i}k_0 d)}{1 - \rho_m^2\exp(\mathrm{i}2k_0 d)}. \tag{2}$$

As for the overall reflection coefficient ρ, we note that the reflected E-field is the superposition of a direct reflection $\rho_m E_0$ from the first mirror, and the transmitted fraction of E_B through the first mirror, albeit after E_B has been phase-shifted by $k_0 d$. We thus write

$$\rho E_0 = \rho_m E_0 + \tau_m E_B\exp(\mathrm{i}k_0 d) = \rho_m E_0 + \tau_m\rho_m E_A\exp(\mathrm{i}2k_0 d) = \left[\rho_m + \frac{\rho_m\tau_m^2\exp(\mathrm{i}2k_0 d)}{1 - \rho_m^2\exp(\mathrm{i}2k_0 d)}\right]E_0$$

$$\rightarrow\; \rho = \frac{[1 - (\rho_m^2 - \tau_m^2)\exp(\mathrm{i}2k_0 d)]\rho_m}{1 - \rho_m^2\exp(\mathrm{i}2k_0 d)}. \tag{3}$$

Digression: For non-absorptive mirrors, it is known that $\rho_m = |\rho_m|e^{\mathrm{i}\varphi_m}$ and $\tau_m = |\tau_m|e^{\mathrm{i}(\varphi_m \pm 90°)}$, so that $\rho_m^2 - \tau_m^2 = (|\rho_m|^2 + |\tau_m|^2)e^{\mathrm{i}2\varphi_m} = e^{\mathrm{i}2\varphi_m}$, where φ_m is a phase angle that depends on the specific structure of the mirror. A similar relation must, therefore, hold for the Fabry-Perot resonator if the mirrors happen to be non-absorptive. To confirm this relation, we write

$$\rho^2 - \tau^2 = \frac{[1 + e^{\mathrm{i}4\varphi_m}\exp(\mathrm{i}4k_0 d) - 2e^{\mathrm{i}2\varphi_m}\exp(\mathrm{i}2k_0 d)]\rho_m^2 - (\rho_m^2 - e^{\mathrm{i}2\varphi_m})^2\exp(\mathrm{i}2k_0 d)}{[1 - \rho_m^2\exp(\mathrm{i}2k_0 d)]^2}$$

$$= \frac{[1 + e^{\mathrm{i}4\varphi_m}\exp(\mathrm{i}4k_0 d)]\rho_m^2 - (\rho_m^4 + e^{\mathrm{i}4\varphi_m})\exp(\mathrm{i}2k_0 d)}{[1 - \rho_m^2\exp(\mathrm{i}2k_0 d)]^2} = \frac{[1 - \rho_m^2\exp(\mathrm{i}2k_0 d)][\rho_m^2 - e^{\mathrm{i}4\varphi_m}\exp(\mathrm{i}2k_0 d)]}{[1 - \rho_m^2\exp(\mathrm{i}2k_0 d)]^2}$$

$$= -\frac{\exp[\mathrm{i}(2k_0 d + 4\varphi_m)]\{1 - |\rho_m|^2\exp[-\mathrm{i}(2k_0 d + 2\varphi_m)]\}}{1 - |\rho_m|^2\exp[\mathrm{i}(2k_0 d + 2\varphi_m)]} = \exp(\mathrm{i}\psi).$$

← bracketed term in the numerator is conjugate of the denominator

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