Problem 1)

a) The dispersion relation $\boldsymbol{k}\cdot\boldsymbol{k}=\mu(\omega)\varepsilon(\omega)(\omega/c)^2$ yields $k\hat{\boldsymbol{\kappa}}\cdot k\hat{\boldsymbol{\kappa}}=k^2=[\omega n(\omega)/c]^2$, resulting in $k=\pm\omega n(\omega)/c$. Therefore, $\boldsymbol{k}'=\pm k_0 n'(\omega)\hat{\boldsymbol{\kappa}}$ and $\boldsymbol{k}''=\pm k_0 n''(\omega)\hat{\boldsymbol{\kappa}}$, where $k_0=\omega/c$ is the wave-number in free space. In what follows, we shall use the plus sign for both $\boldsymbol{k}'$ and $\boldsymbol{k}''$, thus selecting a propagation direction that is along $\hat{\boldsymbol{\kappa}}$ (rather than opposite to $\hat{\boldsymbol{\kappa}}$).

$$\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=\exp\{\mathrm{i}[(k'+\mathrm{i}k'')\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r}-\omega t]\}=\exp(-k''\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r})\times\exp[\mathrm{i}(k'\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r}-\omega t)]$$
$$=\exp(-k_0 n''\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r})\times\exp[\mathrm{i}(k_0 n'\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r}-\omega t)]. \quad (1)$$

b) Let $\boldsymbol{E}_0=E_{x0}\hat{\boldsymbol{x}}+E_{y0}\hat{\boldsymbol{y}}+E_{z0}\hat{\boldsymbol{z}}$ be the E-field amplitude for the plane-wave. In accordance with Maxwell's first equation, we set $\boldsymbol{k}\cdot\boldsymbol{E}_0=k\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{E}_0=0$, which restricts the z-component of the E-field to $E_{z0}=-(E_{x0}\hat{\kappa}_x+E_{y0}\hat{\kappa}_y)/\hat{\kappa}_z$. Aside from this constraint, the remaining components E_{x0} and E_{y0} of the E-field are completely arbitrary. The H-field of the plane-wave is obtained from Maxwell's third equation $\boldsymbol{k}\times\boldsymbol{E}_0=\omega\mu_0\mu(\omega)\boldsymbol{H}_0$. Given that $\mu(\omega)=1.0$ and $\boldsymbol{k}=k\hat{\boldsymbol{\kappa}}$, we find

$$\boldsymbol{H}_0=(\mu_0\omega)^{-1}k\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0=Z_0^{-1}(k/k_0)\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0$$
$$=[n(\omega)/Z_0](\hat{\kappa}_x\hat{\boldsymbol{x}}+\hat{\kappa}_y\hat{\boldsymbol{y}}+\hat{\kappa}_z\hat{\boldsymbol{z}})\times(E_{x0}\hat{\boldsymbol{x}}+E_{y0}\hat{\boldsymbol{y}}+E_{z0}\hat{\boldsymbol{z}})$$
$$=[n(\omega)/Z_0]\big[(\hat{\kappa}_y E_{z0}-\hat{\kappa}_z E_{y0})\hat{\boldsymbol{x}}+(\hat{\kappa}_z E_{x0}-\hat{\kappa}_x E_{z0})\hat{\boldsymbol{y}}+(\hat{\kappa}_x E_{y0}-\hat{\kappa}_y E_{x0})\hat{\boldsymbol{z}}\big]. \quad (2)$$

The H-field amplitude $\boldsymbol{H}_0$ is thus fully specified in terms of $n(\omega)$, the propagation direction $\hat{\boldsymbol{\kappa}}$, and the E-field amplitude $\boldsymbol{E}_0$.

c) $$\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle=\tfrac{1}{2}\mathrm{Re}\{\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}^*(\boldsymbol{r},t)\}$$
$$=\tfrac{1}{2}\mathrm{Re}\{\boldsymbol{E}_0\exp(-k_0 n''\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r})\exp[\mathrm{i}(k_0 n'\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r}-\omega t)]$$
$$\times\boldsymbol{H}_0^*\exp(-k_0 n''\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r})\exp[-\mathrm{i}(k_0 n'\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r}-\omega t)]\}$$
$$=\tfrac{1}{2}\mathrm{Re}\{\boldsymbol{E}_0\times[(n^*/Z_0)\hat{\boldsymbol{\kappa}}\times\boldsymbol{E}_0^*]\}\exp(-2k_0 n''\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r})$$ ← Use $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$
$$=\tfrac{1}{2}\mathrm{Re}\{(n^*/Z_0)[(\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*)\hat{\boldsymbol{\kappa}}-\underbrace{(\boldsymbol{E}_0\cdot\hat{\boldsymbol{\kappa}})}_{0}\boldsymbol{E}_0^*]\}\exp(-2k_0 n''\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r})$$
$$=\tfrac{1}{2}[n'(\omega)/Z_0](\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*)\exp[-2k_0 n''(\omega)\hat{\boldsymbol{\kappa}}\cdot\boldsymbol{r}]\hat{\boldsymbol{\kappa}}. \quad (3)$$

It is seen that the time-averaged energy flows along the $\hat{\boldsymbol{\kappa}}$ direction, that the energy flux is proportional to the real part $n'(\omega)$ of the refractive index, and also proportional to the E-field intensity $\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*=|E_{x0}|^2+|E_{y0}|^2+|E_{z0}|^2$, and that the energy flux declines exponentially along the $\hat{\boldsymbol{\kappa}}$ direction, with an absorption coefficient $\alpha=2k_0 n''(\omega)=(4\pi/\lambda_0)n''(\omega)$.

Note that, while the exponential rate of decay of the field *amplitudes* along $\hat{\boldsymbol{\kappa}}$ is $k_0 n''(\omega)$, the Poynting vector, having contributions from both E and H fields, decays at twice that rate. In the optics literature, the parameter $\alpha=2k_0 n''(\omega)=(4\pi/\lambda_0)n''(\omega)$ is referred to as the "absorption coefficient" of the material medium.

Problem 2)

a) $$\varepsilon(\omega)=1+\chi_e(\omega)=1+\chi_{\rm b}+\frac{\omega_p^2}{\omega_0^2-\omega^2-\mathrm{i}\gamma\omega}=\overbrace{\left[1+\chi_{\rm b}+\frac{\omega_p^2(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\right]}^{\varepsilon'(\omega)}+\mathrm{i}\overbrace{\left[\frac{\omega_p^2\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\right]}^{\varepsilon''(\omega)}.$$

b) $$n(\omega)=\sqrt{\mu(\omega)\varepsilon(\omega)}=\sqrt{\varepsilon'(\omega)+\mathrm{i}\varepsilon''(\omega)}=\sqrt{\varepsilon'(\omega)}\times\sqrt{1+\mathrm{i}(\varepsilon''/\varepsilon')}.$$

c) $$f(x)=f(x_0)+\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=x_0}(x-x_0)+\frac{1}{2!}\left.\frac{\mathrm{d}^2f(x)}{\mathrm{d}x^2}\right|_{x=x_0}(x-x_0)^2+\cdots.$$

When $f(x)=\sqrt{1+x}=(1+x)^{½}$, we will have $\mathrm{d}f(x)/\mathrm{d}x=½(1+x)^{-½}$ and $\mathrm{d}^2f(x)/\mathrm{d}x^2=-¼(1+x)^{-3/2}$. Therefore,

$$f(x)=f(0)+½(1+x)^{-½}\big|_{x=0}x-⅛(1+x)^{-3/2}\big|_{x=0}x^2+\cdots=1+½x-⅛x^2+\cdots.$$

d) $$n(\omega)=\sqrt{\varepsilon'}\times\sqrt{1+\mathrm{i}(\varepsilon''/\varepsilon')}\cong\sqrt{\varepsilon'}[1+½\mathrm{i}(\varepsilon''/\varepsilon')]=\sqrt{\varepsilon'}+\frac{\mathrm{i}\varepsilon''}{2\sqrt{\varepsilon'}}.$$

Thus, to first order in $\varepsilon''/\varepsilon'$, we will have $n'(\omega)\cong\sqrt{\varepsilon'(\omega)}$ and $n''(\omega)\cong\varepsilon''(\omega)/2\sqrt{\varepsilon'(\omega)}$.

Problem 3)

a) The dispersion relation, $k_x^2+k_y^2+k_z^2=(\omega/c)^2\mu(\omega)\varepsilon(\omega)$, yields

$$k_z^{(\rm a)}=\sqrt{(\omega/c)^2\varepsilon_{\rm a}-k_x^2}. \qquad (1)$$

$$k_z^{(\rm b)}=\sqrt{(\omega/c)^2\varepsilon_{\rm b}-k_x^2}. \qquad (2)$$

b) Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t$, yields $\boldsymbol{k}\times\boldsymbol{E}=\omega\mu_0\boldsymbol{H}$. Consequently, the H-field amplitude $\boldsymbol{H}_0$ is expressed in terms of the E-field amplitude $\boldsymbol{E}_0$, the wave-vector $\boldsymbol{k}$, and the oscillation frequency ω as $\boldsymbol{H}_0=(\mu_0\omega)^{-1}(k_x\hat{\boldsymbol{x}}+k_z\hat{\boldsymbol{z}})\times\boldsymbol{E}_0$. We will have

$$\boldsymbol{E}^{(\rm a\pm)}(\boldsymbol{r},t)=E_0^{(\rm a\pm)}\hat{\boldsymbol{y}}\exp[\mathrm{i}(k_xx\pm k_z^{(\rm a)}z-\omega t)]. \qquad (3)$$

$$\boldsymbol{H}^{(\rm a\pm)}(\boldsymbol{r},t)=(\mu_0\omega)^{-1}\big[k_x\hat{\boldsymbol{x}}\pm k_z^{(\rm a)}\hat{\boldsymbol{z}}\big]\times\big[E_0^{(\rm a\pm)}\hat{\boldsymbol{y}}\big]\exp[\mathrm{i}(k_xx\pm k_z^{(\rm a)}z-\omega t)]. \qquad (4)$$

$$\boldsymbol{E}^{(\rm b\pm)}(\boldsymbol{r},t)=E_0^{(\rm b\pm)}\hat{\boldsymbol{y}}\exp[\mathrm{i}(k_xx\pm k_z^{(\rm b)}z-\omega t)]. \qquad (5)$$

$$\boldsymbol{H}^{(\rm b\pm)}(\boldsymbol{r},t)=(\mu_0\omega)^{-1}\big[k_x\hat{\boldsymbol{x}}\pm k_z^{(\rm b)}\hat{\boldsymbol{z}}\big]\times\big[E_0^{(\rm b\pm)}\hat{\boldsymbol{y}}\big]\exp[\mathrm{i}(k_xx\pm k_z^{(\rm b)}z-\omega t)]. \qquad (6)$$

c) Continuity of E_y and H_x at $z=+d/2$:

$$E_y:\quad E_0^{(\rm b+)}\exp\big[½\mathrm{i}k_z^{(\rm b)}d\big]=E_0^{(\rm a+)}\exp\big[½\mathrm{i}k_z^{(\rm a)}d\big]+E_0^{(\rm a-)}\exp\big[-½\mathrm{i}k_z^{(\rm a)}d\big]. \qquad (7)$$

$$H_x:\quad -k_z^{(\rm b)}E_0^{(\rm b+)}\exp\big[½\mathrm{i}k_z^{(\rm b)}d\big]=-k_z^{(\rm a)}E_0^{(\rm a+)}\exp\big[½\mathrm{i}k_z^{(\rm a)}d\big]+k_z^{(\rm a)}E_0^{(\rm a-)}\exp\big[-½\mathrm{i}k_z^{(\rm a)}d\big]. \qquad (8)$$

Dividing Eq.(8) by Eq.(7) eliminates their common factors, as follows:

$$\frac{k_z^{(\rm b)}}{k_z^{(\rm a)}}=\frac{E_0^{(\rm a+)}\exp(\mathrm{i}k_z^{(\rm a)}d)-E_0^{(\rm a-)}}{E_0^{(\rm a+)}\exp(\mathrm{i}k_z^{(\rm a)}d)+E_0^{(\rm a-)}}. \qquad (9)$$

Boundary conditions at $z=-d/2$:

$$E_y: \quad E_0^{(\mathrm{b}-)}\exp[\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{b})}d] = E_0^{(\mathrm{a}+)}\exp[-\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d] + E_0^{(\mathrm{a}-)}\exp[\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d]. \tag{10}$$

$$H_x: \quad k_z^{(\mathrm{b})}E_0^{(\mathrm{b}-)}\exp[\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{b})}d] = -k_z^{(\mathrm{a})}E_0^{(\mathrm{a}+)}\exp[-\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d] + k_z^{(\mathrm{a})}E_0^{(\mathrm{a}-)}\exp[\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d]. \tag{11}$$

Once again, dividing Eq.(11) by Eq.(10) removes their common factors, yielding

$$\frac{k_z^{(\mathrm{b})}}{k_z^{(\mathrm{a})}} = \frac{E_0^{(\mathrm{a}-)}\exp(\mathrm{i}k_z^{(\mathrm{a})}d) - E_0^{(\mathrm{a}+)}}{E_0^{(\mathrm{a}-)}\exp(\mathrm{i}k_z^{(\mathrm{a})}d) + E_0^{(\mathrm{a}+)}}. \tag{12}$$

d) A comparison of Eqs.(9) and (12) now enables one to solve for the ratio $E_0^{(\mathrm{a}+)}/E_0^{(\mathrm{a}-)}$, namely,

$$\frac{E_0^{(\mathrm{a}+)}\exp(\mathrm{i}k_z^{(\mathrm{a})}d) - E_0^{(\mathrm{a}-)}}{E_0^{(\mathrm{a}+)}\exp(\mathrm{i}k_z^{(\mathrm{a})}d) + E_0^{(\mathrm{a}-)}} = \frac{E_0^{(\mathrm{a}-)}\exp(\mathrm{i}k_z^{(\mathrm{a})}d) - E_0^{(\mathrm{a}+)}}{E_0^{(\mathrm{a}-)}\exp(\mathrm{i}k_z^{(\mathrm{a})}d) + E_0^{(\mathrm{a}+)}}$$

$$\rightarrow \quad \cancel{E_0^{(\mathrm{a}+)}E_0^{(\mathrm{a}-)}\exp[2\mathrm{i}k_z^{(\mathrm{a})}}d] + E_0^{(\mathrm{a}+)^2}\exp[\mathrm{i}k_z^{(\mathrm{a})}d] - E_0^{(\mathrm{a}-)^2}\exp[\mathrm{i}k_z^{(\mathrm{a})}d] - \cancel{E_0^{(\mathrm{a}-)}E_0^{(\mathrm{a}+)}}$$

$$= \cancel{E_0^{(\mathrm{a}+)}E_0^{(\mathrm{a}-)}\exp[2\mathrm{i}k_z^{(\mathrm{a})}}d] - E_0^{(\mathrm{a}+)^2}\exp[\mathrm{i}k_z^{(\mathrm{a})}d] + E_0^{(\mathrm{a}-)^2}\exp[\mathrm{i}k_z^{(\mathrm{a})}d] - \cancel{E_0^{(\mathrm{a}-)}E_0^{(\mathrm{a}+)}}$$

$$\rightarrow \quad E_0^{(\mathrm{a}+)^2} = E_0^{(\mathrm{a}-)^2} \quad \rightarrow \quad E_0^{(\mathrm{a}+)} = \pm E_0^{(\mathrm{a}-)}. \tag{13}$$

The solution $E_0^{(\mathrm{a}+)} = E_0^{(\mathrm{a}-)}$ represents the so-called "even modes" of the waveguide, while the solution $E_0^{(\mathrm{a}+)} = -E_0^{(\mathrm{a}-)}$ represents the "odd modes."

e) Substituting the above solutions in either Eq.(9) or Eq.(12) now yields

$$\frac{k_z^{(\mathrm{b})}}{k_z^{(\mathrm{a})}} = \frac{\exp(\mathrm{i}k_z^{(\mathrm{a})}d) \mp 1}{\exp(\mathrm{i}k_z^{(\mathrm{a})}d) \pm 1} = \frac{\exp(\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d) \mp \exp(-\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d)}{\exp(\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d) \pm \exp(-\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d)} = \begin{cases} \dfrac{\sinh(\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d)}{\cosh(\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d)}; & \text{even modes,} \\[2ex] \dfrac{\cosh(\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d)}{\sinh(\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d)}; & \text{odd modes.} \end{cases} \tag{14}$$

Thus the characteristic equation for even modes is $\tanh[\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d] = k_z^{(\mathrm{b})}/k_z^{(\mathrm{a})}$, whereas that for odd modes is $\tanh[\tfrac{1}{2}\mathrm{i}k_z^{(\mathrm{a})}d] = k_z^{(\mathrm{a})}/k_z^{(\mathrm{b})}$. These complex-valued characteristic equations must be solved numerically to yield the allowed k_x values. In general, the equation admits a number of distinct solutions that correspond to the various stable modes of the waveguide. The metallic nature of the cladding in this problem guarantees that the allowed values of k_x will be complex, confirming that the guided modes will attenuate as they propagate forward along the x-axis.

Digression: The solutions obtained are quite general, and can be applied to any pair of values for ε_a and ε_b. For instance, if both ε_a and ε_b are real and positive, with $\varepsilon_\mathrm{a} > \varepsilon_\mathrm{b}$, we will have a dielectric waveguide. In this case, k_x may be real (for guided modes) or complex (for leaky modes)—although it cannot be purely imaginary, because then both $k_z^{(\mathrm{a})}$ and $k_z^{(\mathrm{b})}$ will be real, in which case the characteristic equations $\mathrm{i}\tan[\tfrac{1}{2}k_z^{(\mathrm{a})}d] = k_z^{(\mathrm{b})}/k_z^{(\mathrm{a})}$ and $\mathrm{i}\tan[\tfrac{1}{2}k_z^{(\mathrm{a})}d] = k_z^{(\mathrm{a})}/k_z^{(\mathrm{b})}$ will have no solutions.
