Problem 7.80) a) In the transparent dielectric medium, the k-vector is $\pm(n_0\omega/c)\hat{\boldsymbol{z}}$, and the $\boldsymbol{H}$ field magnitude is n_0E_0/Z_0. Consequently, the incident and reflected fields are given by

$$\boldsymbol{E}^{(\mathrm{i})}(\boldsymbol{r},t)=E_0\exp[-\mathrm{i}(\omega/c)(n_0z+ct)]\hat{\boldsymbol{x}}, \tag{1a}$$

$$\boldsymbol{H}^{(\mathrm{i})}(\boldsymbol{r},t)=-(n_0E_0/Z_0)\exp[-\mathrm{i}(\omega/c)(n_0z+ct)]\hat{\boldsymbol{y}}. \tag{1b}$$

$$\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},t)=\rho E_0\exp[\mathrm{i}(\omega/c)(n_0z-ct)]\hat{\boldsymbol{x}}, \tag{2a}$$

$$\boldsymbol{H}^{(\mathrm{r})}(\boldsymbol{r},t)=(n_0\rho E_0/Z_0)\exp[\mathrm{i}(\omega/c)(n_0z-ct)]\hat{\boldsymbol{y}}. \tag{2b}$$

In the absorptive substrate, the k-vector is complex-valued, that is, $\boldsymbol{k}=-(n+\mathrm{i}\kappa)(\omega/c)\hat{\boldsymbol{z}}$, and the $\boldsymbol{H}$ field amplitude is $(n+\mathrm{i}\kappa)/Z_0$ times that of the $\boldsymbol{E}$ field. Therefore,

$$\boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r},t)=\tau E_0\exp\{-\mathrm{i}(\omega/c)[(n+\mathrm{i}\kappa)z+ct]\}\hat{\boldsymbol{x}}, \tag{3a}$$

$$\boldsymbol{H}^{(\mathrm{t})}(\boldsymbol{r},t)=-(n+\mathrm{i}\kappa)(\tau E_0/Z_0)\exp\{-\mathrm{i}(\omega/c)[(n+\mathrm{i}\kappa)z+ct]\}\hat{\boldsymbol{y}}. \tag{3b}$$

b) Matching the boundary conditions means enforcing the continuity of the $\boldsymbol{E}_\parallel$ and $\boldsymbol{H}_\parallel$ at $z=0$. We will have

$$E_x^{(\mathrm{i})}(z=0^+)+E_x^{(\mathrm{r})}(z=0^+)=E_x^{(\mathrm{t})}(z=0^-)\quad\rightarrow\quad E_0+\rho E_0=\tau E_0, \tag{4a}$$

$$H_y^{(\mathrm{i})}(z=0^+)+H_y^{(\mathrm{r})}(z=0^+)=H_y^{(\mathrm{t})}(z=0^-)\quad\rightarrow\quad n_0E_0-n_0\rho E_0=\tau(n+\mathrm{i}\kappa)E_0. \tag{4b}$$

Solving the above equations for ρ and τ, we find

$$\rho=\frac{n_0-(n+\mathrm{i}\kappa)}{n_0+(n+\mathrm{i}\kappa)}, \tag{5a}$$

$$\tau=1+\rho=\frac{2n_0}{n_0+(n+\mathrm{i}\kappa)}. \tag{5b}$$

c) The time-averaged rate of EM energy flow per unit area per unit time is $\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle=½\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*)$. The corresponding entities for the incident, reflected, and transmitted beams are

$$\begin{aligned}\langle\boldsymbol{S}^{(\mathrm{i})}(\boldsymbol{r},t)\rangle&=-½\mathrm{Re}\{E_0\exp[-\mathrm{i}(\omega/c)(n_0z+ct)]\,(n_0E_0^*/Z_0)\exp[\mathrm{i}(\omega/c)(n_0z+ct)]\}\hat{\boldsymbol{z}}\\&=-½n_0Z_0^{-1}|E_0|^2\hat{\boldsymbol{z}}.\end{aligned} \tag{6}$$

$$\langle\boldsymbol{S}^{(\mathrm{r})}(\boldsymbol{r},t)\rangle=½n_0Z_0^{-1}|\rho E_0|^2\hat{\boldsymbol{z}}. \tag{7}$$

$$\begin{aligned}\langle\boldsymbol{S}^{(\mathrm{t})}(\boldsymbol{r},t)\rangle&=-½\mathrm{Re}\{\tau E_0\exp(-\mathrm{i}(\omega/c)[(n+\mathrm{i}\kappa)z+ct])\\&\qquad\times(n-\mathrm{i}\kappa)\,(\tau^*E_0^*/Z_0)\exp(\mathrm{i}(\omega/c)[(n-\mathrm{i}\kappa)z+ct])\}\hat{\boldsymbol{z}}\\&=-½nZ_0^{-1}|\tau E_0|^2\exp(2\kappa\omega z/c)\,\hat{\boldsymbol{z}}.\end{aligned} \tag{8}$$

d) The energy balance equation at $z=0$ may thus be written as follows:

$$n_0-n_0|\rho|^2=n|\tau|^2\quad\rightarrow\quad|\rho|^2+\left(\frac{n}{n_0}\right)|\tau|^2=1. \tag{9}$$

To confirm the above energy balance equation, note that

$$|\rho|^2 = \rho\rho^* = \frac{(n_0 - n - \mathrm{i}\kappa)(n_0 - n + \mathrm{i}\kappa)}{(n_0 + n + \mathrm{i}\kappa)(n_0 + n - \mathrm{i}\kappa)} = \frac{(n_0 - n)^2 + \kappa^2}{(n_0 + n)^2 + \kappa^2}, \tag{10a}$$

$$|\tau|^2 = \tau\tau^* = \frac{4n_0^2}{(n_0 + n)^2 + \kappa^2}. \tag{10b}$$

Consequently,

$$|\rho|^2 + \left(\frac{n}{n_0}\right)|\tau|^2 = \frac{(n_0 - n)^2 + \kappa^2}{(n_0 + n)^2 + \kappa^2} + \frac{4nn_0}{(n_0 + n)^2 + \kappa^2} = \frac{(n_0 + n)^2 + \kappa^2}{(n_0 + n)^2 + \kappa^2} = 1. \tag{11}$$

The energy absorbed in the substrate is thus seen to be precisely equal to the difference between the incident and reflected energies.
