Problem 7.79) a) The various plane-waves are all linearly-polarized along the x-axis, with their magnetic fields aligned with either $\hat{\boldsymbol{y}}$ or $-\hat{\boldsymbol{y}}$. The k-vector is either along $\hat{\boldsymbol{z}}$ or $-\hat{\boldsymbol{z}}$, its magnitude being $k_0 = \omega/c$ in air, $n_0 k_0$ in the dielectric layer, and $(n + \mathrm{i}\kappa)k_0$ in the metallic substrate. The $\boldsymbol{E}$ and $\boldsymbol{H}$ fields in the three regions are

Incident beam: $\boldsymbol{E}^{(\mathrm{i})}(\boldsymbol{r},t) = E_0\hat{\boldsymbol{x}}\exp[-\mathrm{i}(\omega/c)(z+ct)],$

$\boldsymbol{H}^{(\mathrm{i})}(\boldsymbol{r},t) = -(E_0/Z_0)\hat{\boldsymbol{y}}\exp[-\mathrm{i}(\omega/c)(z+ct)].$

Reflected beam: $\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},t) = \rho E_0\hat{\boldsymbol{x}}\exp[\mathrm{i}(\omega/c)(z-ct)],$

$\boldsymbol{H}^{(\mathrm{r})}(\boldsymbol{r},t) = \rho(E_0/Z_0)\hat{\boldsymbol{y}}\exp[\mathrm{i}(\omega/c)(z-ct)].$

Inside dielectric: $\boldsymbol{E}^{(\mathrm{A})}(\boldsymbol{r},t) = E_A\hat{\boldsymbol{x}}\exp[-\mathrm{i}(\omega/c)(n_0 z+ct)],$

$\boldsymbol{H}^{(\mathrm{A})}(\boldsymbol{r},t) = -(n_0 E_A/Z_0)\hat{\boldsymbol{y}}\exp[-\mathrm{i}(\omega/c)(n_0 z+ct)].$

$\boldsymbol{E}^{(\mathrm{B})}(\boldsymbol{r},t) = E_B\hat{\boldsymbol{x}}\exp[\mathrm{i}(\omega/c)(n_0 z-ct)],$

$\boldsymbol{H}^{(\mathrm{B})}(\boldsymbol{r},t) = (n_0 E_B/Z_0)\hat{\boldsymbol{y}}\exp[\mathrm{i}(\omega/c)(n_0 z-ct)].$

Inside metal: $\boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r},t) = \tau E_0\hat{\boldsymbol{x}}\exp\{-\mathrm{i}(\omega/c)[(n+\mathrm{i}\kappa)z+ct]\},$

$\boldsymbol{H}^{(\mathrm{t})}(\boldsymbol{r},t) = -[(n+\mathrm{i}\kappa)\tau E_0/Z_0]\hat{\boldsymbol{y}}\exp\{-\mathrm{i}(\omega/c)[(n+\mathrm{i}\kappa)z+ct]\}.$

b) Matching the boundary conditions at the air-glass interface ($z = 0$) yields

Continuity of $E_\parallel$: $E_0 + \rho E_0 = E_A + E_B \quad \rightarrow \quad (1+\rho)E_0 = E_A + E_B.$

Continuity of $H_\parallel$: $-\frac{E_0}{Z_0} + \frac{\rho E_0}{Z_0} = -\frac{n_0 E_A}{Z_0} + \frac{n_0 E_B}{Z_0} \quad \rightarrow \quad (1-\rho)E_0 = n_0(E_A - E_B).$

Dividing the above equations eliminates E_0, yielding ρ in terms of $\rho_1 = (1-n_0)/(1+n_0)$ and E_B/E_A, as follows:

$$\frac{1-\rho}{1+\rho} = n_0\frac{1-(E_B/E_A)}{1+(E_B/E_A)} \quad \rightarrow \quad \rho = \frac{[(1-n_0)/(1+n_0)] + (E_B/E_A)}{1 + [(1-n_0)/(1+n_0)](E_B/E_A)} = \frac{\rho_1 + (E_B/E_A)}{1+\rho_1(E_B/E_A)}.$$

Matching the boundary conditions at the glass-metal interface ($z = -d_0$) yields

Continuity of $E_\parallel$: $E_A\exp(\mathrm{i}\omega n_0 d_0/c) + E_B\exp(-\mathrm{i}\omega n_0 d_0/c) = \tau E_0\exp[\mathrm{i}(\omega/c)(n+\mathrm{i}\kappa)d_0].$

Continuity of $H_\parallel$: $-(n_0 E_A/Z_0)\exp(\mathrm{i}\omega n_0 d_0/c) + (n_0 E_B/Z_0)\exp(-\mathrm{i}\omega n_0 d_0/c)$

$= -[(n+\mathrm{i}\kappa)\tau E_0/Z_0]\exp[\mathrm{i}(\omega/c)(n+\mathrm{i}\kappa)d_0].$

Dividing the above equations eliminates the transmission coefficient τ, yielding an expression for E_B/E_A in terms of $\rho_2 = [n_0 - (n+\mathrm{i}\kappa)]/[n_0 + (n+\mathrm{i}\kappa)] = |\rho_2|\exp(\mathrm{i}\varphi_2)$ and the round-trip phase $\varphi_0 = 2\omega n_0 d_0/c = 4\pi n_0 d_0/\lambda_0$, namely,

$$\frac{E_A \exp(\mathrm{i}\omega n_0 d_0/c) - E_B \exp(-\mathrm{i}\omega n_0 d_0/c)}{E_A \exp(\mathrm{i}\omega n_0 d_0/c) + E_B \exp(-\mathrm{i}\omega n_0 d_0/c)} = \frac{n+\mathrm{i}\kappa}{n_0} \quad \rightarrow \quad \frac{\exp(\mathrm{i}\varphi_0) - (E_B/E_A)}{\exp(\mathrm{i}\varphi_0) + (E_B/E_A)} = (n+\mathrm{i}\kappa)/n_0$$

$$\rightarrow \quad \frac{E_B}{E_A} = \frac{n_0 - (n+\mathrm{i}\kappa)}{n_0 + (n+\mathrm{i}\kappa)} \exp(\mathrm{i}\varphi_0) = \rho_2 \exp(\mathrm{i}\varphi_0) = |\rho_2| \exp[\mathrm{i}(\varphi_2 + \varphi_0)].$$

c) Substitution for E_B/E_A from the above equation into an earlier expression for ρ now yields

$$\rho = \frac{\rho_1 + |\rho_2| \exp[\mathrm{i}(\varphi_2+\varphi_0)]}{1 + \rho_1 |\rho_2| \exp[\mathrm{i}(\varphi_2+\varphi_0)]}.$$

The reflectance of the coated metallic surface is obtained by squaring the magnitude of the above Fresnel reflection coefficient, that is,

$$R = |\rho|^2 = \rho\rho^* = \frac{\rho_1^2 + |\rho_2|^2 + 2\rho_1|\rho_2|\cos(\varphi_0+\varphi_2)}{1 + \rho_1^2|\rho_2|^2 + 2\rho_1|\rho_2|\cos(\varphi_0+\varphi_2)}.$$

d) The reflectance R is a function of the round-trip phase $\varphi_0 = 4\pi n_0 d_0/\lambda_0$, which may be adjusted by varying the thickness d_0 of the dielectric layer. The maximum and minimum values of R are determined by setting its derivative with respect to φ_0 equal to zero, that is,

$$\frac{dR}{d\varphi_0} = -\frac{2\rho_1|\rho_2|(1-\rho_1^2)(1-|\rho_2|^2)\sin(\varphi_0+\varphi_2)}{\left[1 + \rho_1^2|\rho_2|^2 + 2\rho_1|\rho_2|\cos(\varphi_0+\varphi_2)\right]^2} = 0 \quad \rightarrow \quad \sin(\varphi_0+\varphi_2) = 0.$$

Considering that $\rho_1 < 0$, the reflectance reaches a minimum when $\cos(\varphi_0 + \varphi_2) = 1$, that is,

$$R_{\min} = \left(\frac{\rho_1 + |\rho_2|}{1 + \rho_1|\rho_2|}\right)^2.$$

The maximum reflectance is reached when $\cos(\varphi_0 + \varphi_2) = -1$, that is,

$$R_{\max} = \left(\frac{\rho_1 - |\rho_2|}{1 - \rho_1|\rho_2|}\right)^2.$$
