Problem 7.78) The k-vector in free space is aligned with $\hat{\boldsymbol{z}}$ and has magnitude $k_0 = \omega_0/c$. The $\boldsymbol{E}$ and $\boldsymbol{H}$ field amplitudes are $|E_0|\exp(\mathrm{i}\varphi_0)\,\hat{\boldsymbol{x}}$ and $Z_0^{-1}|E_0|\exp(\mathrm{i}\varphi_0)\,\hat{\boldsymbol{y}}$. We thus have

a) $$\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\{E_0\hat{\boldsymbol{x}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_0 t)]\} = |E_0|\cos(k_0 z-\omega_0 t+\varphi_0)\,\hat{\boldsymbol{x}}, \tag{1a}$$

$$\boldsymbol{H}(\boldsymbol{r},t) = \mathrm{Re}\{H_0\hat{\boldsymbol{y}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_0 t)]\} = Z_0^{-1}|E_0|\cos(k_0 z-\omega_0 t+\varphi_0)\,\hat{\boldsymbol{y}}. \tag{1b}$$

b) The rate of flow of electromagnetic (EM) energy per unit area per unit time is given by the Poynting vector, as follows:

$$\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t) = Z_0^{-1}|E_0|^2\cos^2(k_0 z-\omega_0 t+\varphi_0)\,\hat{\boldsymbol{z}}$$
$$= \tfrac{|E_0|^2}{2Z_0}\{1+\cos[2(k_0 z-\omega_0 t+\varphi_0)]\}\hat{\boldsymbol{z}}. \tag{2}$$

At $t=t_0$, the values of the Poynting vector at P_1 and P_2 are $S(0,0,z_1,t_0)$ and $S(0,0,z_2,t_0)$, respectively. Therefore, the difference between the rates of energy inflow and outflow is

$$S_1 - S_2 = ½(|E_0|^2/Z_0)\{\cos[2(k_0 z_1-\omega_0 t_0+\varphi_0)] - \cos[2(k_0 z_2-\omega_0 t_0+\varphi_0)]\}$$
$$= -(|E_0|^2/Z_0)\sin[k_0(z_1-z_2)]\sin[k_0(z_1+z_2)-2\omega_0 t_0+2\varphi_0]. \tag{3}$$

The above difference between S_1 and S_2 is seen to vanish if the distance between z_1 and z_2 happens to be $z_2 - z_1 = m\pi/k_0 = m\pi c/\omega_0 = m\lambda_0/2$, with m being an arbitrary integer.

c) The energy-density of the EM field in free space is $\mathcal{E}(\boldsymbol{r},t) = ½\varepsilon_0 E^2 + ½\mu_0 H^2$. Therefore, for the plane-wave described in part (a), we have

$$\mathcal{E}(\boldsymbol{r},t) = ½(\varepsilon_0+\mu_0/Z_0^2)|E_0|^2\cos^2(k_0 z-\omega_0 t+\varphi_0) = \varepsilon_0|E_0|^2\cos^2(k_0 z-\omega_0 t+\varphi_0)$$
$$= ½\varepsilon_0|E_0|^2\{1+\cos[2(k_0 z-\omega_0 t+\varphi_0)]\}. \tag{4}$$

The integrated energy-density between z_1 and z_2 is thus equal to the energy (per unit cross-sectional area) contained in the region between P_1 and P_2, namely,

$$\int_{z_1}^{z_2}\mathcal{E}(\boldsymbol{r},t)dz = ½\varepsilon_0|E_0|^2\int_{z_1}^{z_2}\{1+\cos[2(k_0 z-\omega_0 t+\varphi_0)]\}dz$$
$$= ½\varepsilon_0|E_0|^2(z_2-z_1)$$
$$+¼\varepsilon_0|E_0|^2k_0^{-1}\{\sin[2(k_0 z_2-\omega_0 t+\varphi_0)] - \sin[2(k_0 z_1-\omega_0 t+\varphi_0)]\}$$
$$= ½\varepsilon_0|E_0|^2(z_2-z_1)$$
$$-½(\varepsilon_0 c/\omega_0)|E_0|^2\sin[k_0(z_1-z_2)]\cos[k_0(z_1+z_2)-2\omega_0 t+2\varphi_0]. \tag{5}$$

Differentiating the above expression with respect to time now yields the time-rate-of-change of the stored EM energy in the region between P_1 and P_2 (per unit cross-sectional area), as follows:

$$\frac{d}{dt}\int_{z_1}^{z_2}\mathcal{E}(\boldsymbol{r},t)dz = -\varepsilon_0 c|E_0|^2\sin[k_0(z_1-z_2)]\sin[k_0(z_1+z_2)-2\omega_0 t+2\varphi_0]. \tag{6}$$

Given that $\varepsilon_0 c = 1/Z_0$, a comparison of Eq.(3) with Eq.(6) reveals that the difference between S_1 and S_2 is fully accounted for in terms of the time-rate-of-change of the stored EM energy in the region between P_1 and P_2.