Problem 7.76)

In the air, where the refractive index is essentially equal to 1.0, we have $\boldsymbol{k} = k_z\hat{\boldsymbol{z}} = \pm(\omega/c)\hat{\boldsymbol{z}}$, and the E-field to H-field amplitude ratio is $E_0/H_0 = Z_0 = \sqrt{\mu_0/\varepsilon_0}$.

a) Incident beam: $\boldsymbol{E}^{(\mathrm{i})}(\boldsymbol{r},t) = E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}[-(\omega/c)z-\omega t]\}$, (1a)

$\boldsymbol{H}^{(\mathrm{i})}(\boldsymbol{r},t) = -(E_0/Z_0)\hat{\boldsymbol{y}}\exp\{\mathrm{i}[-(\omega/c)z-\omega t]\}$. (1b)

Reflected beam: $\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},t) = \rho E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}[(\omega/c)z-\omega t]\}$, (2a)

$\boldsymbol{H}^{(\mathrm{r})}(\boldsymbol{r},t) = (\rho E_0/Z_0)\hat{\boldsymbol{y}}\exp\{\mathrm{i}[(\omega/c)z-\omega t]\}$. (2b)

Transmitted beam: $\boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r},t) = \tau E_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}[-(\omega/c)z-\omega t]\}$, (3a)

$\boldsymbol{H}^{(\mathrm{t})}(\boldsymbol{r},t) = -(\tau E_0/Z_0)\hat{\boldsymbol{y}}\exp\{\mathrm{i}[-(\omega/c)z-\omega t]\}$. (3b)

Inside the slab: $\boldsymbol{E}^{(\mathrm{A})}(\boldsymbol{r},t) = aE_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}[-(n\omega/c)z-\omega t]\}$, (4a)

$\boldsymbol{H}^{(\mathrm{A})}(\boldsymbol{r},t) = -(anE_0/Z_0)\hat{\boldsymbol{y}}\exp\{\mathrm{i}[-(n\omega/c)z-\omega t]\}$. (4b)

$\boldsymbol{E}^{(\mathrm{B})}(\boldsymbol{r},t) = bE_0\hat{\boldsymbol{x}}\exp\{\mathrm{i}[(n\omega/c)z-\omega t]\}$, (5a)

$\boldsymbol{H}^{(\mathrm{B})}(\boldsymbol{r},t) = (bnE_0/Z_0)\hat{\boldsymbol{y}}\exp\{\mathrm{i}[(n\omega/c)z-\omega t]\}$. (5b)

b) At the top of the slab, where $z=0$, we have

Continuity of $\boldsymbol{E}_\parallel$: $E_0 + \rho E_0 = aE_0 + bE_0$. (6a)

Continuity of $\boldsymbol{H}_\parallel$: $Z_0^{-1}(-E_0+\rho E_0) = Z_0^{-1}(-anE_0 + bnE_0)$. (6b)

At the bottom of the slab, where $z=-d$, we have

Continuity of $\boldsymbol{E}_\parallel$: $aE_0\exp(\mathrm{i}n\omega d/c) + bE_0\exp(-\mathrm{i}n\omega d/c) = \tau E_0\exp(\mathrm{i}\omega d/c)$. (7a)

Continuity of $\boldsymbol{H}_\parallel$:

$$Z_0^{-1}[-anE_0\exp(\mathrm{i}n\omega d/c) + bnE_0\exp(-\mathrm{i}n\omega d/c)] = -Z_0^{-1}\tau E_0\exp(\mathrm{i}\omega d/c). \quad (7b)$$

c) Equations (6) and (7) may now be solved to determine the coefficients a, b, ρ, τ, as follows:

$$1+\rho = a+b, \quad (8a)$$

$$1-\rho = (a-b)n, \quad (8b)$$

$$a\exp(\mathrm{i}n\omega d/c) + b\exp(-\mathrm{i}n\omega d/c) = \tau\exp(\mathrm{i}\omega d/c), \quad (8c)$$

$$an\exp(\mathrm{i}n\omega d/c) - bn\exp(-\mathrm{i}n\omega d/c) = \tau\exp(\mathrm{i}\omega d/c). \quad (8d)$$

From the above equations we find

$$\frac{1-\rho}{1+\rho} = n\left[\frac{1-(b/a)}{1+(b/a)}\right], \tag{9a}$$

$$\frac{b}{a} = \left(\frac{n-1}{n+1}\right)\exp(\mathrm{i}2n\omega d/c). \tag{9b}$$

Substituting from Eq.(9b) into Eq.(9a), then solving for ρ, we arrive at

$$\frac{1-\rho}{1+\rho} = n\left[\frac{1-[(n-1)/(n+1)]\exp(\mathrm{i}2n\omega d/c)}{1+[(n-1)/(n+1)]\exp(\mathrm{i}2n\omega d/c)}\right] \quad \rightarrow \quad \rho = \frac{[(n-1)/(n+1)][1-\exp(\mathrm{i}2n\omega d/c)]}{[(n-1)/(n+1)]^2\exp(\mathrm{i}2n\omega d/c)-1}. \tag{10}$$

Subsequently, the Fresnel transmission coefficient τ is obtained from Eq.(8c) with the aid of Eqs.(8a) and (8b), as follows:

$$\tau\exp(\mathrm{i}\omega d/c) = (a+b)\cos(n\omega d/c) + \mathrm{i}(a-b)\sin(n\omega d/c)$$

$$\rightarrow \quad \tau = [(1+\rho)\cos(n\omega d/c) + \mathrm{i}n^{-1}(1-\rho)\sin(n\omega d/c)]\exp(-\mathrm{i}\omega d/c). \tag{11}$$

In the above expressions for ρ and τ, one may replace $(n\omega d/c)$ with $(2\pi nd/\lambda_0)$, where λ_0 is the incident beam's vacuum wavelength.

d) With reference to Eq.(10), the reflectance will be zero when $\exp(\mathrm{i}2n\omega d/c) = 1$. This happens when $n\omega d/c$ becomes an integer-multiple of π, or, equivalently, when $2nd/\lambda_0$ becomes an integer. Thus, when the thickness of the slab is an integer-multiple of half-wavelength within the dielectric, i.e., $\lambda_0/2n$, the reflectance of the slab precisely equals zero.

e) From symmetry of Eq.(10), the maximum reflectance must occur halfway between adjacent minima. Thus when the thickness d is an odd-multiple of a quarter-wavelength within the dielectric, i.e., $\lambda_0/4n$, we will have $\exp(\mathrm{i}2n\omega d/c) = -1$, at which point the reflectance will be a maximum, that is,

$$R_{\max} = |\rho|^2 = 4\left(\frac{n-1}{n+1}\right)^2 \Big/ \left[1+\left(\frac{n-1}{n+1}\right)^2\right]^2. \tag{12}$$

Note: To see the symmetry of Eq.(10), start the phase-angle $(2n\omega d/c)$ at an integer-multiple of 2π, and change it by $\pm\varphi$. You will find that the corresponding values of ρ are conjugates of each other, and that, therefore, the corresponding values of $R = |\rho|^2$ are identical.
