Problem 7.75) a) In the free-space region, the incident k-vector is $\boldsymbol{k}^{(i)} = (\omega/c)(\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}})$. The E and H fields may be written in terms of $\boldsymbol{k}^{(\mathrm{i})}$, ω, and the E-field amplitude E_0, as follows:

$$\boldsymbol{E}^{(\mathrm{i})}(\boldsymbol{r},t) = \mathrm{Re}\{E_0(\cos\theta\,\hat{\boldsymbol{x}} - \sin\theta\,\hat{\boldsymbol{z}})\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{r} - \omega t)]\},$$

$$\boldsymbol{H}^{(\mathrm{i})}(\boldsymbol{r},t) = \mathrm{Re}\{Z_0^{-1}E_0\hat{\boldsymbol{y}}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{i})}\cdot\boldsymbol{r} - \omega t)]\}.$$

b) For the reflected beam, the k-vector is $\boldsymbol{k}^{(\mathrm{r})} = (\omega/c)(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}})$, and the E and H fields, expressed as functions of $\boldsymbol{k}^{(\mathrm{r})}$, ω, the Fresnel reflection coefficient ρ_p, and the incident E-field amplitude E_0, are

$$\boldsymbol{E}^{(\mathrm{r})}(\boldsymbol{r},t) = \mathrm{Re}\{\rho_p E_0(\cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{z}})\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{r})}\cdot\boldsymbol{r} - \omega t)]\},$$

$$\boldsymbol{H}^{(\mathrm{r})}(\boldsymbol{r},t) = -\,\mathrm{Re}\{Z_0^{-1}\rho_p E_0\hat{\boldsymbol{y}}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{r})}\cdot\boldsymbol{r} - \omega t)]\}.$$

c) For the transmitted beam, the k-vector is $\boldsymbol{k}^{(\mathrm{t})} = (\omega/c)[\sin\theta\,\hat{\boldsymbol{x}} + \sqrt{\varepsilon(\omega) - \sin^2\theta}\,\hat{\boldsymbol{z}}]$. This is derived from the continuity of k_x across the interface, and from the dispersion relation of the plasma, namely, $k_x^2 + k_z^2 = (\omega/c)^2\mu(\omega)\varepsilon(\omega)$. The E and H fields, written in terms of $\boldsymbol{k}^{(\mathrm{t})}$, ω, the Fresnel transmission coefficient τ_p, and the incident E-field amplitude E_0, are

$$\boldsymbol{E}^{(\mathrm{t})}(\boldsymbol{r},t) = \mathrm{Re}\left\{\tau_p E_0\cos\theta\left(\hat{\boldsymbol{x}} - \frac{\sin\theta}{\sqrt{\varepsilon(\omega)-\sin^2\theta}}\hat{\boldsymbol{z}}\right)\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{r} - \omega t)]\right\},$$

$$\boldsymbol{H}^{(\mathrm{t})}(\boldsymbol{r},t) = \mathrm{Re}\left\{\frac{\tau_p\varepsilon(\omega)E_0\cos\theta}{Z_0\sqrt{\varepsilon(\omega)-\sin^2\theta}}\hat{\boldsymbol{y}}\exp[\mathrm{i}(\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{r} - \omega t)]\right\}.$$

In deriving the above expressions, we used the constraints imposed by Maxwell's 1st and 3rd equations, namely, $\boldsymbol{k}^{(\mathrm{t})}\cdot\boldsymbol{E}^{(\mathrm{t})} = k_x^{(\mathrm{t})}E_x^{(\mathrm{t})} + k_z^{(\mathrm{t})}E_z^{(\mathrm{t})} = 0$ and $\boldsymbol{k}^{(\mathrm{t})}\times\boldsymbol{E}^{(\mathrm{t})} = \mu_0\mu(\omega)\omega\boldsymbol{H}^{(\mathrm{t})}$.

d) The tangential components $E_x^{(\mathrm{i})}, E_x^{(\mathrm{r})}, E_x^{(\mathrm{t})}$ of the E-field must satisfy the continuity condition at the interface, as do the tangential components $H_y^{(\mathrm{i})}, H_y^{(\mathrm{r})}, H_y^{(\mathrm{t})}$ of the H-field. Therefore,

$$\boldsymbol{E}_\parallel \text{ continuity:}\quad E_0\cos\theta + \rho_p E_0\cos\theta = \tau_p E_0\cos\theta \quad\rightarrow\quad 1 + \rho_p = \tau_p.$$

$$\boldsymbol{H}_\parallel \text{ continuity:}\quad Z_0^{-1}E_0 - Z_0^{-1}\rho_p E_0 = \frac{\tau_p\varepsilon(\omega)E_0\cos\theta}{Z_0\sqrt{\varepsilon(\omega)-\sin^2\theta}} \quad\rightarrow\quad 1 - \rho_p = \frac{\tau_p\varepsilon(\omega)\cos\theta}{\sqrt{\varepsilon(\omega)-\sin^2\theta}}.$$

Solving the above equations, we find $\rho_p = \dfrac{\sqrt{\varepsilon(\omega)-\sin^2\theta}-\varepsilon(\omega)\cos\theta}{\sqrt{\varepsilon(\omega)-\sin^2\theta}+\varepsilon(\omega)\cos\theta}$ and $\tau_p = \dfrac{2\sqrt{\varepsilon(\omega)-\sin^2\theta}}{\sqrt{\varepsilon(\omega)-\sin^2\theta}+\varepsilon(\omega)\cos\theta}$.

e) Since $\varepsilon(\omega)$ is real-valued and negative, ρ_p may be written as follows:

$$\rho_p = \frac{\mathrm{i}\sqrt{|\varepsilon(\omega)| + \sin^2\theta} + |\varepsilon(\omega)|\cos\theta}{\mathrm{i}\sqrt{|\varepsilon(\omega)| + \sin^2\theta} - |\varepsilon(\omega)|\cos\theta}$$

Thus ρ_p is seen to be the ratio of a complex number to its conjugate, which has a magnitude of 1. Since $|\rho_p| = 1$, the reflectivity is 100%. This does not contradict the existence of electromagnetic waves within the plasma, because the time-averaged Poynting vector of the plane-wave inside the plasma, like that of an evanescent wave, has a vanishing z-component.