Problem 7.74)

a) $$\boldsymbol{k} = k_0\hat{\boldsymbol{z}} = (\omega/c)\hat{\boldsymbol{z}}.$$

b) The beam is linearly-polarized if either $E_{x0} = 0$ or $E_{y0} = 0$ or $\varphi_{x0} = \varphi_{y0}$ or $\varphi_{x0} = \varphi_{y0} \pm \pi$. The beam is circularly-polarized if $E_{x0} = E_{y0}$ *and* $\varphi_{x0} - \varphi_{y0} = \pm\pi/2$. Under all other circumstances, the beam will be elliptically-polarized.

c) Starting with the assumption that the amplitude and phase of the *H*-field components are (H_{x0}, ψ_{x0}) and (H_{y0}, ψ_{y0}), we write

$$\boldsymbol{H}(\boldsymbol{r}, t) = H_{x0}\cos(k_0 z - \omega t + \psi_{x0})\,\hat{\boldsymbol{x}} + H_{y0}\cos(k_0 z - \omega t + \psi_{y0})\,\hat{\boldsymbol{y}}.$$

Maxwell's 3rd equation then yields

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \quad\rightarrow\quad -\frac{\partial E_y}{\partial z}\hat{\boldsymbol{x}} + \frac{\partial E_x}{\partial z}\hat{\boldsymbol{y}} = -\mu_0\left(\frac{\partial H_x}{\partial t}\hat{\boldsymbol{x}} + \frac{\partial H_y}{\partial t}\hat{\boldsymbol{y}}\right).$$

Consequently,

$$-\frac{\partial E_y}{\partial z} = -\mu_0\frac{\partial H_x}{\partial t} \rightarrow k_0 E_{y0}\sin(k_0 z - \omega t + \varphi_{y0}) = -\mu_0 H_{x0}\omega\sin(k_0 z - \omega t + \psi_{x0})$$
$$\rightarrow (\omega/c)E_{y0}\sin(k_0 z - \omega t + \varphi_{y0}) = -\mu_0 H_{x0}\omega\sin(k_0 z - \omega t + \psi_{x0})$$
$$\rightarrow H_{x0} = -E_{y0}/(\mu_0 c) = -E_{y0}/Z_0 \quad\text{and}\quad \psi_{x0} = \varphi_{y0}.$$

Similarly,

$$\frac{\partial E_x}{\partial z} = -\mu_0\frac{\partial H_y}{\partial t} \rightarrow -k_0 E_{x0}\sin(k_0 z - \omega t + \varphi_{x0}) = -\mu_0 H_{y0}\omega\sin(k_0 z - \omega t + \psi_{y0})$$
$$\rightarrow (\omega/c)E_{x0}\sin(k_0 z - \omega t + \varphi_{x0}) = \mu_0 H_{y0}\omega\sin(k_0 z - \omega t + \psi_{y0})$$
$$\rightarrow H_{y0} = E_{x0}/(\mu_0 c) = E_{x0}/Z_0 \quad\text{and}\quad \psi_{y0} = \varphi_{x0}.$$

d) Direct multiplication of the *E*-field into the *H*-field obtained in part (c) now yields

$$\boldsymbol{S}(\boldsymbol{r}, t) = Z_0^{-1}\left[E_{x0}\cos(k_0 z - \omega t + \varphi_{x0})\,\hat{\boldsymbol{x}} + E_{y0}\cos(k_0 z - \omega t + \varphi_{y0})\,\hat{\boldsymbol{y}}\right]$$
$$\times\left[-E_{y0}\cos(k_0 z - \omega t + \varphi_{y0})\,\hat{\boldsymbol{x}} + E_{x0}\cos(k_0 z - \omega t + \varphi_{x0})\,\hat{\boldsymbol{y}}\right]$$
$$= Z_0^{-1}\left[E_{x0}^2\cos^2(k_0 z - \omega t + \varphi_{x0}) + E_{y0}^2\cos^2(k_0 z - \omega t + \varphi_{y0})\right]\hat{\boldsymbol{z}}.$$

The Poynting vector $\boldsymbol{S}(\boldsymbol{r}, t)$ is the rate of flow of electromagnetic energy per unit area per unit time, evaluated at the point $\boldsymbol{r}$ in space and at the instant t of time. It *must* satisfy the energy continuity equation at *all* points $\boldsymbol{r}$ in space at *all* instants t in time.

e) For circular-polarization, we have $E_{x0} = E_{y0}$ and $\varphi_{x0} = \varphi_{y0} \pm \pi/2$. Therefore,

$$\boldsymbol{S}(\boldsymbol{r}, t) = Z_0^{-1}E_{x0}^2[\cos^2(k_0 z - \omega t + \varphi_{x0}) + \sin^2(k_0 z - \omega t + \varphi_{x0})]\hat{\boldsymbol{z}} = Z_0^{-1}E_{x0}^2\,\hat{\boldsymbol{z}}.$$

Clearly, the above expression is independent of z and t. The electromagnetic energy thus flows uniformly and at the constant rate of E_{x0}^2/Z_0 along the z-axis

f) For a linearly-polarized beam, we will have

$$\boldsymbol{S}(\boldsymbol{r},t) = Z_0^{-1}\left(E_{x0}^2 + E_{y0}^2\right)\cos^2(k_0 z - \omega t + \varphi_{x0})\,\hat{\boldsymbol{z}}.$$

The above $\boldsymbol{S}$ obviously varies with both z and t. This means that at any given time, say, $t = t_0$, the energy crossing a plane perpendicular to the z-axis at z_1 is different from the energy crossing another perpendicular plane at z_2. Conservation of energy is not violated, however, because, unlike the case of circular-polarization, the energy stored in the E and H fields in the region between z_1 and z_2 is not constant in this case. Recall that Poynting's theorem in free-space requires that $\boldsymbol{\nabla}\cdot\boldsymbol{S} + \partial(\tfrac{1}{2}\varepsilon_0\boldsymbol{E}\cdot\boldsymbol{E} + \tfrac{1}{2}\mu_0\boldsymbol{H}\cdot\boldsymbol{H})/\partial t = 0$. Consequently, the difference between the energy entering at $z = z_1$ and the energy leaving at $z = z_2$ is given to (or taken away from) the energy stored in the E and H fields in the space between z_1 and z_2.