Problem 7.72) a) The E-field energy-density is $\frac{1}{2}\varepsilon_0|\boldsymbol{E}|^2$. Since the E-field oscillates with frequency ω_o, time-averaging yields the average E-field energy-density as $\frac{1}{4}\varepsilon_0 E_0^2$. Multiplying this into the volume cTA of the pulse, we obtain the E-field energy of the pulse as $\frac{1}{4}\varepsilon_0 cTAE_0^2$. Similarly, the amplitude of the H-field of the light is $H_0 = E_0/Z_0$. Since the time-averaged magnetic energy density in vacuum is given by $\frac{1}{4}\mu_0 H_0^2 = \frac{1}{4}\varepsilon_0 E_0^2$, the magnetic energy of the pulse is equal to its electric energy. The total energy is thus given by $\frac{1}{2}\varepsilon_0 cTAE_0^2$.

Alternatively, we may compute the time-averaged Poynting vector as follows:

$$<\boldsymbol{S}> = \tfrac{1}{2}\text{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = \tfrac{1}{2}E_0 H_0\hat{\boldsymbol{z}} = [E_0^2/(2Z_o)]\hat{\boldsymbol{z}}.$$

This is the rate of flow of energy per unit area per unit time at any given cross-section of the light pulse. Multiplication with A and T then yields the total energy of the pulse as $ATE_0^2/(2Z_o)$. Considering that $\varepsilon_o c = 1/Z_o$, the two expressions obtained above for the total pulse energy are exactly the same.

b) The reflected pulse has the same frequency ω_o and the same wavelength $\lambda_o = 2\pi c/\omega_o$ as the incident pulse. Its polarization state is also linear and in the same direction as the incident polarization. The pulse duration and cross-sectional area remain T and A, respectively. The only things that change are the field amplitudes E_o and H_o, which are multiplied by the Fresnel reflection coefficient $\rho = (1-n)/(1+n)$. The reflected pulse energy is therefore given by ρ^2 times the incident pulse energy, that is, $(1-n)^2 ATE_0^2/[2Z_o(1+n)^2]$.

c) The Fresnel transmission coefficient at the entrance facet of the glass slab is $\tau = 1+\rho = 2/(1+n)$. This means that the E-field amplitude inside the glass slab is $2E_0/(1+n)$. The H-field amplitude is n times the E-field amplitude divided by Z_o, that is, $H_0 = 2nE_0/[Z_o(1+n)]$. Therefore, the z-component of the Poynting vector inside the slab is $<S_z> = 2nE_0^2/[Z_o(1+n)^2]$. Since the pulse duration T and the cross-sectional area A inside the slab remain the same as outside, the total energy of the transmitted pulse is $2nATE_0^2/[Z_o(1+n)^2]$. Other properties of the transmitted pulse are: frequency $= \omega_o$, wavelength $\lambda = \lambda_o/n$, pulse length $= cT/n$, polarization state = linear and in the same direction as the incident pulse.

Alternatively, one may evaluate the energy densities of the E and H fields separately, then add them together. We find

$$\text{Time-averaged } E\text{-field energy density} = \tfrac{1}{4}\varepsilon_o\varepsilon(\tau E_0)^2 = \tfrac{1}{4}\varepsilon_o n^2[2E_0/(1+n)]^2 = \varepsilon_o n^2 E_0^2/(1+n)^2.$$

$$\text{Time-averaged } H\text{-field energy density} = \tfrac{1}{4}\mu_o(n\tau E_0/Z_o)^2 = \tfrac{1}{4}\mu_o\{2nE_0/[Z_o(1+n)]\}^2 = \varepsilon_o n^2 E_0^2/(1+n)^2.$$

Adding the above energy densities, then multiplying by the pulse volume cAT/n yields the same result as before, namely, transmitted pulse energy $= 2\varepsilon_o c\, nATE_0^2/(1+n)^2$.

d) Reflected plus transmitted pulse energy =

$$(1-n)^2 ATE_0^2/[2Z_o(1+n)^2] + 2nATE_0^2/[Z_o(1+n)^2] = ATE_0^2/(2Z_o).$$

This is the same as the incident pulse energy; therefore, energy is conserved.