Problem 7.68)

a) $$\boldsymbol{E}(z,t) = E_o\hat{\boldsymbol{x}}\sin(kz-\omega t) + E_o\hat{\boldsymbol{x}}\sin(kz+\omega t) = 2E_o\hat{\boldsymbol{x}}\sin(kz)\cos(\omega t).$$

$$\boldsymbol{H}(z,t) = Z_o^{-1}E_o\hat{\boldsymbol{y}}[\sin(kz-\omega t) - \sin(kz+\omega t)] = -2Z_o^{-1}E_o\hat{\boldsymbol{y}}\cos(kz)\sin(\omega t).$$

b) Denoting the electromagnetic energy-density at point z and instant t by $\mathcal{E}(z,t)$, we will have

$$\mathcal{E}(z,t) = \tfrac{1}{2}\varepsilon_o\boldsymbol{E}^2(z,t) + \tfrac{1}{2}\mu_o\boldsymbol{H}^2(z,t) = 2\varepsilon_o E_o^2\sin^2(kz)\cos^2(\omega t) + 2\mu_o Z_o^{-2}E_o^2\cos^2(kz)\sin^2(\omega t)$$

$$= 2\varepsilon_o E_o^2\sin^2(kz)(1-\sin^2\omega t) + 2\varepsilon_o E_o^2\cos^2(kz)\sin^2(\omega t)$$

$$= 2\varepsilon_o E_o^2[\sin^2(kz) + \cos(2kz)\sin^2(\omega t)].$$

Integration of the above energy-density over z from 0 to L yields $L/2$ for $\sin^2(kz)$ and zero for $\cos(2kz)$. The total energy contained in the cavity thus turns out to be $\varepsilon_o E_o^2 AL$, independent of time. The number of photons is now found to be $\mathcal{N} = \varepsilon_o E_o^2 AL/(\hbar\omega)$. At any given moment, one-half of these photons may be said to be propagating from left to right, while the remaining half travel from right to left.

c) At $z=0$, the magnetic field at the mirror surface is $\boldsymbol{H}(z=0^+,t) = -2Z_o^{-1}E_o\hat{\boldsymbol{y}}\sin(\omega t)$. Since the fields inside a perfect conductor are always zero, Maxwell's boundary conditions require that $\boldsymbol{J}_s(z=0,t) = 2Z_o^{-1}E_o\hat{\boldsymbol{x}}\sin(\omega t)$. The effective magnetic field acting on this surface-current is the *average* of the H-fields immediately in front of and immediately beneath the surface, that is, $\boldsymbol{H}_{\text{eff}}(z=0,t) = -Z_o^{-1}E_o\hat{\boldsymbol{y}}\sin(\omega t)$. The Lorentz force-density exerted by this field on the surface current $\boldsymbol{J}_s$ is, therefore, going to be

$$\boldsymbol{J}_s(z=0,t)\times\mu_o\boldsymbol{H}_{\text{eff}}(z=0,t) = 2Z_o^{-1}E_o\sin(\omega t)\hat{\boldsymbol{x}}\times[-\mu_o Z_o^{-1}E_o\sin(\omega t)\hat{\boldsymbol{y}}] = -2\varepsilon_o E_o^2\sin^2(\omega t)\hat{\boldsymbol{z}}.$$

Time-averaging and integration over the surface area of the mirror yields a total force equal to $\langle\boldsymbol{F}(z=0,t)\rangle = -\varepsilon_o E_o^2 A\hat{\boldsymbol{z}}$. A similar procedure (or an argument from symmetry) yields the force on the mirror located at $z=L$ as $\langle\boldsymbol{F}(z=L,t)\rangle = \varepsilon_o E_o^2 A\hat{\boldsymbol{z}}$.

d) The work done by radiation pressure in moving the mirror a distance Δz adds to the kinetic energy of the mirror. Since the mirror is stationary at first, the work done in moving it from $z=L$ to $z=L+\Delta z$ must be equal to its final kinetic energy, that is, $\tfrac{1}{2}MV^2 = F_z\Delta z = \varepsilon_o E_o^2 A\,\Delta z$.

e) For the forward propagating wave inside the cavity, the Poynting vector is given by

$$\boldsymbol{S}(z,t) = \boldsymbol{E}(z,t)\times\boldsymbol{H}(z,t) = E_o\sin(kz-\omega t)\hat{\boldsymbol{x}}\times Z_o^{-1}E_o\sin(kz-\omega t)\hat{\boldsymbol{y}} = Z_o^{-1}E_o^2\sin^2(kz-\omega t)\hat{\boldsymbol{z}}.$$

The electromagnetic momentum density is given by $\boldsymbol{S}(z,t)/c^2$. Integration over the volume of the cavity thus yields the total momentum content of the forward-propagating wave as $\tfrac{1}{2}\varepsilon_o E_o^2 AL/c$. Upon reflection from the mirror located at $z=L$, *twice* this momentum will be transferred to the mirror during the time interval $\frac{1}{2}\Delta t = L/c$, which is the time it takes for the entire forward-propagating wave within the cavity to bounce off the front mirror. By this time, the other plane-wave has turned around and is now propagating in the forward direction, so, during the next $\frac{1}{2}\Delta t$ interval, another transfer of momentum (from the electromagnetic field to the front mirror) takes place. The total momentum acquired by the front mirror between $t = t_o$ and $t = t_o + 2L/c$ is, therefore, given by $MV = 2\varepsilon_0 E_o^2 AL/c = \varepsilon_0 E_o^2 A\Delta t$. This result is also consistent with Newton's law, $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$, when applied to the front mirror under the influence of the electromagnetic force $<\boldsymbol{F}(z=L,t)> = \varepsilon_o E_o^2 A\hat{\boldsymbol{z}}$. Alternatively, one could obtain the momentum of the mirror by multiplying the total number of photons $\mathcal{N}$, computed in part (b), with *twice* the momentum of each photon, $\hbar\omega/c$, thus arriving at $MV = 2\mathcal{N}\hbar\omega/c = 2\varepsilon_0 E_o^2 AL/c$.

f) Since $L = N\lambda/2 = \pi cN/\omega$, we have $\mathrm{d}L/\mathrm{d}\omega = -\pi cN/\omega^2 = -L/\omega$. Therefore, $\Delta z/L = -\Delta\omega/\omega$.

g) In part (d) we found that $\frac{1}{2}MV^2 = \varepsilon_0 E_o^2 A\Delta z$. Writing $\frac{1}{2}MV^2 = \frac{1}{2}(MV)V$ and substituting for the momentum MV the result obtained in part (e), namely, $MV = \varepsilon_0 E_o^2 A\Delta t$, we find $\Delta z = \frac{1}{2}V\Delta t$. This should not come as a surprise, however, considering that over the short time interval Δt, the radiation pressure exerts a constant force on the front mirror, resulting in a constant acceleration a. Since $\Delta z = \frac{1}{2}a(\Delta t)^2$ and $V = a\Delta t$, it is obvious that Δz must be equal to $\frac{1}{2}V\Delta t$. None of the results obtained thus far require the restriction of Δt to the specific value $2L/c$; in other words, we expect to find the same results for *any* sufficiently small Δt.

Next, we substitute the above Δz into the expression for $\Delta\omega/\omega$ obtained in part (f). We find

$$\Delta\omega/\omega = -\Delta z/L = -\tfrac{1}{2}V\Delta t/L = -V/c.$$

This is the formula for the Doppler shift upon reflection from a flat mirror moving at the constant velocity $\frac{1}{2}V\hat{\boldsymbol{z}}$, which, in the system under consideration, is the *average* mirror velocity during the time interval $\Delta t = 2L/c$. What is special about $\Delta t = 2L/c$ is that it is the time interval during which each and every photon inside the cavity gets exactly one chance to bounce off the front mirror. Without this restriction, one does *not* arrive at the Doppler relation between $\Delta\omega$ and the (average) mirror velocity. The red-shift (or cooling down) of the photons is a direct consequence of their transfer of energy and momentum to the free-standing mirror.

At much later times, when $t \gg t_o$, the free mirror will have an initial velocity V in the beginning of each cycle, reaching $V + \delta V$ after a time interval $\Delta t = 2L/c$. (L is now the length of the cavity in the beginning of the cycle.) The increase in the kinetic energy of the mirror will then be $F_z\Delta z = MV\delta V$, whereas the change in the mirror's momentum will be $F_z\Delta t = M\delta V$. Consequently, $\Delta z = V\Delta t$ and $\Delta\omega/\omega = -\Delta z/L = -2V/c$, which is the Doppler shift upon reflection from a mirror moving at an average velocity $V + \frac{1}{2}\delta V$, provided, of course, that δV is negligible compared to V.