Problem 7.66)

a) The plane-waves' E- and H-fields have the following general form:

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_\text{o}\exp[\text{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_\text{o}t)],$$

$$\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}_\text{o}\exp[\text{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_\text{o}t)].$$

For the incident (i), reflected (r), and transmitted (t) beams we have

$$\boldsymbol{k}^{(\text{i})} = (n_1\omega_\text{o}/c)(\sin\theta\hat{\boldsymbol{y}} - \cos\theta\hat{\boldsymbol{z}}),$$
$$\boldsymbol{k}^{(\text{r})} = (n_1\omega_\text{o}/c)(\sin\theta\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}),$$
$$\boldsymbol{k}^{(\text{t})} = (n_2\omega_\text{o}/c)(\sin\theta'\hat{\boldsymbol{y}} - \cos\theta'\hat{\boldsymbol{z}}).$$

In what follows, we shall use Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$, to relate the H-field to the E-field via the unit vector $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$ along the propagation direction, namely,

$$\boldsymbol{H}_\text{o} = \boldsymbol{k}\times\boldsymbol{E}_\text{o}/(\mu_\text{o}\omega_\text{o}) = (n/Z_\text{o})\hat{\boldsymbol{k}}\times\boldsymbol{E}_\text{o}.$$

Defining the Fresnel reflection and transmission coefficients for p- and s-light as ρ_p, ρ_s, τ_p, and τ_s, we write

$$\boldsymbol{E}_\text{o}^{(\text{i})} = E_s^{(\text{i})}\hat{\boldsymbol{x}} + E_p^{(\text{i})}(\cos\theta\hat{\boldsymbol{y}} + \sin\theta\hat{\boldsymbol{z}}),$$
$$\boldsymbol{H}_\text{o}^{(\text{i})} = (n_1/Z_\text{o})(\sin\theta\hat{\boldsymbol{y}} - \cos\theta\hat{\boldsymbol{z}})\times\boldsymbol{E}_\text{o}^{(\text{i})} = (n_1/Z_\text{o})[E_p^{(\text{i})}\hat{\boldsymbol{x}} - E_s^{(\text{i})}(\cos\theta\hat{\boldsymbol{y}} + \sin\theta\hat{\boldsymbol{z}})].$$

$$\boldsymbol{E}_\text{o}^{(\text{r})} = \rho_s E_s^{(\text{i})}\hat{\boldsymbol{x}} + \rho_p E_p^{(\text{i})}(\cos\theta\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{z}}),$$
$$\boldsymbol{H}_\text{o}^{(\text{r})} = (n_1/Z_\text{o})(\sin\theta\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}})\times\boldsymbol{E}_\text{o}^{(\text{r})} = (n_1/Z_\text{o})[-\rho_p E_p^{(\text{i})}\hat{\boldsymbol{x}} + \rho_s E_s^{(\text{i})}(\cos\theta\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{z}})].$$

$$\boldsymbol{E}_\text{o}^{(\text{t})} = \tau_s E_s^{(\text{i})}\hat{\boldsymbol{x}} + \tau_p E_p^{(\text{i})}(\cos\theta'\hat{\boldsymbol{y}} + \sin\theta'\hat{\boldsymbol{z}}),$$
$$\boldsymbol{H}_\text{o}^{(\text{t})} = (n_2/Z_\text{o})(\sin\theta'\hat{\boldsymbol{y}} - \cos\theta'\hat{\boldsymbol{z}})\times\boldsymbol{E}_\text{o}^{(\text{t})} = (n_2/Z_\text{o})[\tau_p E_p^{(\text{i})}\hat{\boldsymbol{x}} - \tau_s E_s^{(\text{i})}(\cos\theta'\hat{\boldsymbol{y}} + \sin\theta'\hat{\boldsymbol{z}})].$$

Note that the Snell's law requirement, $k_y^{(\text{i})} = k_y^{(\text{r})} = k_y^{(\text{t})}$, is readily satisfied for the reflected beam by setting $\theta^{(\text{r})} = \theta^{(\text{i})} = \theta$, whereas for the transmitted beam we must have $n_1\sin\theta = n_2\sin\theta'$. Also, within the transmission medium, $(k_y^2 + k_z^2)^{(\text{t})} = (n_2\omega_\text{o}/c)^2$ results in the following relation: $k_z^{(\text{t})} = -(n_2\omega_\text{o}/c)\cos\theta' = -(n_2\omega_\text{o}/c)\sqrt{1-\sin^2\theta'}$. As long as $\sin\theta'$ is below unity, the argument of the square root will be non-negative and, therefore, the sign of the square root will be positive (by convention). However, when $n_1\sin\theta > n_2$, the square root becomes imaginary, necessitating a choice between + and – for its sign. In the geometry chosen for this problem, we must choose $\cos\theta' = \text{i}\sqrt{\sin^2\theta' - 1} = \text{i}\sqrt{(n_1^2\sin^2\theta/n_2^2) - 1}$, to ensure that the evanescent wave inside the transmission medium decays exponentially away from the interface. Note also that τ_p is defined slightly differently here than in Chapter 7; here τ_p is the transmission coefficient for E_p, not E_y.

b) To satisfy the boundary conditions we equate the tangential components of the E- and H-fields across the interface. We will have

Continuity of E_x: $E_s^{(\mathrm{i})} + \rho_s E_s^{(\mathrm{i})} = \tau_s E_s^{(\mathrm{i})}$ $\rightarrow$ $1+\rho_s = \tau_s$,

Continuity of E_y: $E_p^{(\mathrm{i})}\cos\theta + \rho_p E_p^{(\mathrm{i})}\cos\theta = \tau_p E_p^{(\mathrm{i})}\cos\theta'$ $\rightarrow$ $(1+\rho_p)\cos\theta = \tau_p\cos\theta'$,

Continuity of H_x: $n_1 E_p^{(\mathrm{i})} - n_1\rho_p E_p^{(\mathrm{i})} = n_2\tau_p E_p^{(\mathrm{i})}$ $\rightarrow$ $n_1(1-\rho_p) = n_2\tau_p$,

Continuity of H_y: $-n_1 E_s^{(\mathrm{i})}\cos\theta + n_1\rho_s E_s^{(\mathrm{i})}\cos\theta = -n_2\tau_s E_s^{(\mathrm{i})}\cos\theta'$ $\rightarrow$ $n_1(1-\rho_s)\cos\theta = n_2\tau_s\cos\theta'$.

The 1st and 4th of the above equations then yield

$$\rho_s = \frac{n_1\cos\theta - n_2\cos\theta'}{n_1\cos\theta + n_2\cos\theta'}, \qquad \tau_s = \frac{2n_1\cos\theta}{n_1\cos\theta + n_2\cos\theta'}.$$

Similarly, from the 2nd and 3rd equations we find

$$\rho_p = \frac{n_1\cos\theta' - n_2\cos\theta}{n_1\cos\theta' + n_2\cos\theta}, \qquad \tau_p = \frac{2n_1\cos\theta}{n_1\cos\theta' + n_2\cos\theta}.$$

c) When $\rho_p = 0$ we have $n_1\cos\theta' = n_2\cos\theta$, from which, after some algebraic manipulations, we obtain $\tan\theta = n_2/n_1$ and $\tan\theta' = n_1/n_2$. This incidence angle at which the reflectivity of p-polarized light vanishes is known as Brewster's angle, θ_B. There is no Brewster's angle for s-light.

d) When $\cos\theta'$ becomes imaginary, the magnitudes of both ρ_p and ρ_s become unity, that is, $|\rho_p| = |\rho_s| = 1$. This is because these Fresnel coefficients assume the form $(a - \mathrm{i}b)/(a + \mathrm{i}b)$, which, as the ratio of two complex numbers of equal magnitude, has a magnitude of 1. As mentioned earlier, for $\cos\theta'$ to become imaginary, the incidence angle must exceed a certain critical angle, i.e., $n_1\sin\theta > n_2$, which happens when $n_1 > n_2$ and $\theta > \theta_{\mathrm{crit}} = \arcsin(n_2/n_1)$. These conditions apply to p-light and s-light alike. Beyond the critical angle θ_{crit}, both p- and s-polarized incident beams get totally reflected at the interface, although the phase of the reflection coefficient ρ_p differs from that of ρ_s at any given incidence angle θ.
