Problem 7.64)

a) Lorenz Gauge: $\boldsymbol{\nabla}\cdot\boldsymbol{A}+\dfrac{1}{c^2}\dfrac{\partial\psi}{\partial t}=0 \;\rightarrow\; \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{A}_o-(\mathrm{i}\omega/c^2)\psi_o=0 \;\rightarrow\; \boldsymbol{k}\cdot\boldsymbol{A}_o=(\omega/c^2)\psi_o.$

b) $\boldsymbol{E}=-\boldsymbol{\nabla}\psi-\dfrac{\partial \boldsymbol{A}}{\partial t} \;\rightarrow\; \boldsymbol{E}(\boldsymbol{r},t)=(-\mathrm{i}\boldsymbol{k}\psi_o+\mathrm{i}\omega\boldsymbol{A}_o)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$

$$\rightarrow\; \boldsymbol{E}_o=\mathrm{i}(\omega\boldsymbol{A}_o-\boldsymbol{k}\psi_o) \;\rightarrow\; \boldsymbol{E}_o=\mathrm{i}\omega[\boldsymbol{A}_o-(c/\omega)^2(\boldsymbol{k}\cdot\boldsymbol{A}_o)\boldsymbol{k}].$$

$$\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A} \;\rightarrow\; \boldsymbol{B}(\boldsymbol{r},t)=\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_o\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)] \;\rightarrow\; \boldsymbol{B}_o=\mu_o\boldsymbol{H}_o=\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_o.$$

c) In free space, $\rho_{\text{free}}=0$, $\boldsymbol{J}_{\text{free}}=0$, $\boldsymbol{P}=0$, and $\boldsymbol{M}=0$. Consequently, $\boldsymbol{D}=\varepsilon_o\boldsymbol{E}$ and $\boldsymbol{B}=\mu_o\boldsymbol{H}$. Maxwell's equations thus become

i) $\boldsymbol{\nabla}\cdot\varepsilon_o\boldsymbol{E}=0 \;\rightarrow\; \mathrm{i}\varepsilon_o\boldsymbol{k}\cdot\boldsymbol{E}_o\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]=0 \;\rightarrow\; \boldsymbol{k}\cdot\boldsymbol{E}_o=0 \;\rightarrow\; \boldsymbol{k}\cdot(\omega\boldsymbol{A}_o-\boldsymbol{k}\psi_o)=0$

$$\rightarrow\; \omega\boldsymbol{k}\cdot\boldsymbol{A}_o-\boldsymbol{k}^2\psi_o=0 \;\rightarrow\; (\omega/c)^2\psi_o-\boldsymbol{k}^2\psi_o=0 \;\rightarrow\; \text{Either } \psi_o=0 \text{ or } \boldsymbol{k}^2=(\omega/c)^2.$$

ii) $\boldsymbol{\nabla}\times\boldsymbol{H}=\dfrac{\partial\varepsilon_o\boldsymbol{E}}{\partial t} \;\rightarrow\; \boldsymbol{\nabla}\times\boldsymbol{B}=\boldsymbol{\nabla}\times\mu_o\boldsymbol{H}=\dfrac{\partial\boldsymbol{E}}{c^2\partial t} \;\rightarrow\; \mathrm{i}\boldsymbol{k}\times(\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_o)=-(\mathrm{i}\omega/c^2)\boldsymbol{E}_o$

$$\rightarrow\; (\boldsymbol{k}\cdot\boldsymbol{A}_o)\boldsymbol{k}-\boldsymbol{k}^2\boldsymbol{A}_o=-(\omega/c^2)(\omega\boldsymbol{A}_o-\boldsymbol{k}\psi_o)$$

$$\rightarrow\; [\boldsymbol{k}^2-(\omega/c)^2]\boldsymbol{A}_o=[\boldsymbol{k}\cdot\boldsymbol{A}_o-(\omega/c^2)\psi_o]\boldsymbol{k}=0 \;\rightarrow\; \boldsymbol{k}^2=(\omega/c)^2.$$

iii) $\boldsymbol{\nabla}\times\boldsymbol{E}=-\dfrac{\partial\boldsymbol{B}}{\partial t} \;\rightarrow\; \mathrm{i}\boldsymbol{k}\times[\mathrm{i}(\omega\boldsymbol{A}_o-\boldsymbol{k}\psi_o)]=\mathrm{i}\omega(\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_o)$

$$\rightarrow\; \omega\boldsymbol{k}\times\boldsymbol{A}_o-(\boldsymbol{k}\times\boldsymbol{k})\psi_o=\omega\boldsymbol{k}\times\boldsymbol{A}_o \;\rightarrow\; (\boldsymbol{k}\times\boldsymbol{k})\psi_o=0 \;\rightarrow\; 0=0.$$

iv) $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \;\rightarrow\; \mathrm{i}\boldsymbol{k}\cdot(\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_o)=0 \;\rightarrow\; (\boldsymbol{k}\times\boldsymbol{k})\cdot\boldsymbol{A}_o=0 \;\rightarrow\; 0=0.$

Thus, the only constraint on the parameters, aside from the Lorenz gauge $\boldsymbol{k}\cdot\boldsymbol{A}_o=(\omega/c^2)\psi_o$, is $\boldsymbol{k}^2=(\omega/c)^2$.

d) The plane-wave is inhomogeneous (or evanescent) when the imaginary component of $\boldsymbol{k}$ is non-zero. The constraint $\boldsymbol{k}^2=(\omega/c)^2$ thus yields

$$\boldsymbol{k}^2=(\omega/c)^2 \;\rightarrow\; (\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')\cdot(\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}'')=(\omega/c)^2 \;\rightarrow\; \boldsymbol{k}'^2-\boldsymbol{k}''^2+2\mathrm{i}\,\boldsymbol{k}'\cdot\boldsymbol{k}''=(\omega/c)^2$$

$$\rightarrow\; \boldsymbol{k}'^2-\boldsymbol{k}''^2=(\omega/c)^2 \quad \text{and} \quad \boldsymbol{k}'\cdot\boldsymbol{k}''=0.$$

For the plane-wave to be evanescent, it is thus necessary for $\boldsymbol{k}'$ and $\boldsymbol{k}''$ to be orthogonal to each other. It is also necessary for $|\boldsymbol{k}'|$ to be greater than ω/c, so that $|\boldsymbol{k}''|$ will be real-valued.

e) When $\boldsymbol{k}$ is a real-valued vector, i.e., when $\boldsymbol{k}''=0$, the plane-wave will be homogeneous. Both $\boldsymbol{E}_o$ and $\boldsymbol{B}_o$ will then be proportional to the transverse component $\boldsymbol{A}_{o\perp}=\boldsymbol{A}_o-(c/\omega)^2(\boldsymbol{k}\cdot\boldsymbol{A}_o)\boldsymbol{k}$ of $\boldsymbol{A}_o$, with $\boldsymbol{B}_o$ rotated around the k-vector by 90º. The plane-wave will be linearly polarized if the real and imaginary parts of this transverse vector potential, namely, $\boldsymbol{A}'_{o\perp}$ and $\boldsymbol{A}''_{o\perp}$, happen to be parallel to each other, or if one of them (either $\boldsymbol{A}'_{o\perp}$ or $\boldsymbol{A}''_{o\perp}$) vanishes.

f) Again, the plane-wave is homogeneous when $\boldsymbol{k}''=0$. As before, $\boldsymbol{E}_o$ and $\boldsymbol{B}_o$ will be proportional to $\boldsymbol{A}_{o\perp}$, with $\boldsymbol{B}_o$ rotated around $\boldsymbol{k}$ by 90º. The plane-wave will be circularly polarized if $\boldsymbol{A}'_{o\perp}$ and $\boldsymbol{A}''_{o\perp}$ happen to be equal in magnitude and perpendicular to each other.

g) The time-averaged rate of flow of electromagnetic energy is given by the time-averaged Poynting vector, that is,

$$
\begin{aligned}
<\boldsymbol{S}(\boldsymbol{r},t)> &= \tfrac{1}{2}\operatorname{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = \tfrac{1}{2}\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\operatorname{Re}(\boldsymbol{E}_o\times\boldsymbol{H}_o^*)\\
&= \tfrac{1}{2}\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\operatorname{Re}\{(\omega\boldsymbol{A}_o-\boldsymbol{k}\psi_o)\times\mu_o^{-1}\boldsymbol{k}^*\times\boldsymbol{A}_o^*\}\\
&= \frac{\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})}{2\mu_o}\operatorname{Re}\{[(\boldsymbol{A}_o\cdot\boldsymbol{A}_o^*)\boldsymbol{k}^* - (\boldsymbol{k}^*\cdot\boldsymbol{A}_o)\boldsymbol{A}_o^*]\omega + [(\boldsymbol{k}\cdot\boldsymbol{k}^*)\boldsymbol{A}_o^* - (\boldsymbol{k}\cdot\boldsymbol{A}_o^*)\boldsymbol{k}^*]\psi_o\}.
\end{aligned}
$$