Problem 7.62) a) The expression for the E-field is $\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{\text{o}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$. The k-vector is, in general, complex-valued, meaning that $\boldsymbol{k}=\boldsymbol{k}'+\mathrm{i}\boldsymbol{k}''$. The propagation direction is given by $\boldsymbol{k}'$, while $\boldsymbol{k}''$ specifies the direction along which the beam is attenuated (whenever $k''\neq 0$). The E-field amplitude is given by the complex-valued vector $\boldsymbol{E}_{\text{o}}=\boldsymbol{E}_{\text{o}}'+\mathrm{i}\boldsymbol{E}_{\text{o}}''$. In the MKSA system of units, $\boldsymbol{E}$ and $\boldsymbol{E}_{\text{o}}$ have units of *volt/meter*, $\boldsymbol{k}$ has units of m^{-1}, and ω has units of sec^{-1} (or *radians/sec*).

b) If the real-valued vectors $\boldsymbol{E}_{\text{o}}'$ and $\boldsymbol{E}_{\text{o}}''$ are aligned with each other, or if one of them happens to be zero, then the E-field is said to be linearly-polarized. When both $\boldsymbol{E}_{\text{o}}'$ and $\boldsymbol{E}_{\text{o}}''$ are non-zero and also have different orientations in space, the E-field is circularly or elliptically polarized. (For circular polarization, $\boldsymbol{E}_{\text{o}}'$ and $\boldsymbol{E}_{\text{o}}''$ must have equal magnitudes and be perpendicular to each other.)

c) The expression for the H-field is $\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{\text{o}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$. The H-field amplitude is given by the complex-valued vector $\boldsymbol{H}_{\text{o}}=\boldsymbol{H}_{\text{o}}'+\mathrm{i}\boldsymbol{H}_{\text{o}}''$. In the MKSA system of units, $\boldsymbol{H}$ and $\boldsymbol{H}_{\text{o}}$ have units of *ampere/meter*.

d) In the absence of $\boldsymbol{P}(\boldsymbol{r},t)$ and $\rho_{\text{free}}(\boldsymbol{r},t)$, we will have $\boldsymbol{D}(\boldsymbol{r},t)=\varepsilon_{\text{o}}\boldsymbol{E}(\boldsymbol{r},t)$, and Maxwell's 1st equation reduces to $\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t)=0$. Substituting the E-field distribution of part (a) in this equation then yields $\boldsymbol{k}\cdot\boldsymbol{E}_{\text{o}}=0$, which is the constraint imposed on $\boldsymbol{k}$ and $\boldsymbol{E}_{\text{o}}$ by Maxwell's 1st equation.

e) Using the E- and H-field distributions given in (a) and (c), Maxwell's 2nd equation yields: $\boldsymbol{k}\times\boldsymbol{H}_{\text{o}}=-\omega\varepsilon_{\text{o}}\boldsymbol{E}_{\text{o}}$, which is the only constraint imposed by the 2nd equation on $\boldsymbol{k}$, ω, $\boldsymbol{E}_{\text{o}}$, and $\boldsymbol{H}_{\text{o}}$.

f) Using the E- and H-field distributions given in (a) and (c), Maxwell's 3rd equation yields: $\boldsymbol{k}\times\boldsymbol{E}_{\text{o}}=\omega\mu_{\text{o}}\boldsymbol{H}_{\text{o}}$, which is the only constraint imposed by the 3rd equation on $\boldsymbol{k}$, ω, $\boldsymbol{E}_{\text{o}}$, and $\boldsymbol{H}_{\text{o}}$.

g) In the absence of $\boldsymbol{M}(\boldsymbol{r},t)$ we will have $\boldsymbol{B}(\boldsymbol{r},t)=\mu_{\text{o}}\boldsymbol{H}(\boldsymbol{r},t)$, and Maxwell's 4th equation reduces to $\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r},t)=0$. Substituting the H-field distribution of part (c) in this equation then yields $\boldsymbol{k}\cdot\boldsymbol{H}_{\text{o}}=0$, which is the sole constraint imposed on $\boldsymbol{k}$ and $\boldsymbol{H}_{\text{o}}$ by Maxwell's 4th equation.

h) In part (f) we found $\boldsymbol{H}_{\text{o}}=(\omega\mu_{\text{o}})^{-1}\boldsymbol{k}\times\boldsymbol{E}_{\text{o}}$. Substituting this expression for $\boldsymbol{H}_{\text{o}}$ into the constraint obtained in part (e) yields: $\boldsymbol{k}\times[(\omega\mu_{\text{o}})^{-1}\boldsymbol{k}\times\boldsymbol{E}_{\text{o}}]=-\omega\varepsilon_{\text{o}}\boldsymbol{E}_{\text{o}}$. Using the vector identity $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$ we write the preceding equation as $(\boldsymbol{k}\cdot\boldsymbol{E}_{\text{o}})\boldsymbol{k}-(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{E}_{\text{o}}=-\mu_{\text{o}}\varepsilon_{\text{o}}\omega^2\boldsymbol{E}_{\text{o}}$. From part (d) we know that $\boldsymbol{k}\cdot\boldsymbol{E}_{\text{o}}=0$; therefore, $(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{E}_{\text{o}}=\mu_{\text{o}}\varepsilon_{\text{o}}\omega^2\boldsymbol{E}_{\text{o}}$. Dropping $\boldsymbol{E}_{\text{o}}$ from both sides of this equation and using the fact that $\mu_{\text{o}}\varepsilon_{\text{o}}=1/c^2$ now yields $k^2=(\omega/c)^2$, which is the desired dispersion relation.