Problem 7.60)

By definition $\rho_{\text{p}} = E_{x0}^{(\text{r})}/E_{x0}^{(\text{i})}$. We shall also invoke the generalized form of Snell's law, $k_x^{(\text{r})} = k_x^{(\text{i})}$, and the dispersion relation $k_x^2 + k_z^2 = (\omega/c)^2 n_0^2(\omega)$.

a)
$$\boldsymbol{k}^{(\text{i})} = n_0(\omega)(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}}).$$
$$\boldsymbol{E}_{\text{p}}^{(\text{i})}(\boldsymbol{r},t) = \left(E_{x0}^{(\text{i})}\hat{\boldsymbol{x}} + E_{z0}^{(\text{i})}\hat{\boldsymbol{z}}\right)\exp\left[\text{i}\left(\boldsymbol{k}^{(\text{i})}\cdot\boldsymbol{r} - \omega t\right)\right].$$

From Maxwell's 1st equation: $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0 \quad\rightarrow\quad \boldsymbol{k}^{(\text{i})}\cdot\boldsymbol{E}_{\text{p}}^{(\text{i})} = 0 \quad\rightarrow\quad E_{z0}^{(\text{i})} = (\tan\theta)E_{x0}^{(\text{i})}.$

From Maxwell's 3rd equation: $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t \quad\rightarrow\quad \boldsymbol{k}^{(\text{i})}\times\boldsymbol{E}_{\text{p}}^{(\text{i})} = \mu_0\omega\boldsymbol{H}_0^{(\text{i})}$

$$\begin{aligned}\rightarrow\quad \boldsymbol{H}_0^{(\text{i})} &= Z_0^{-1}n_0(\omega)(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}})\times(E_{x0}^{(\text{i})}\hat{\boldsymbol{x}} + E_{z0}^{(\text{i})}\hat{\boldsymbol{z}})\\ &= -Z_0^{-1}n_0(\omega)[\sin\theta\,E_{z0}^{(\text{i})} + \cos\theta\,E_{x0}^{(\text{i})}]\hat{\boldsymbol{y}}\\ &= -Z_0^{-1}n_0(\omega)E_{x_0}^{(\text{i})}\hat{\boldsymbol{y}}/\cos\theta\\ &= -Z_0^{-1}n_0(\omega)E_{\text{p}}^{(\text{i})}\hat{\boldsymbol{y}}.\end{aligned}$$

Consequently, $\boldsymbol{H}^{(\text{i})}(\boldsymbol{r},t) = \boldsymbol{H}_0^{(\text{i})}\exp\left[\text{i}\left(\boldsymbol{k}^{(\text{i})}\cdot\boldsymbol{r} - \omega t\right)\right].$

Applying similar procedures to the reflected beam, we find

$$\boldsymbol{k}^{(\text{r})} = n_0(\omega)(\omega/c)(\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}}).$$
$$\boldsymbol{E}_{\text{p}}^{(\text{r})}(\boldsymbol{r},t) = \left(E_{x0}^{(\text{r})}\hat{\boldsymbol{x}} + E_{z0}^{(\text{r})}\hat{\boldsymbol{z}}\right)\exp\left[\text{i}\left(\boldsymbol{k}^{(\text{r})}\cdot\boldsymbol{r} - \omega t\right)\right].$$

From Maxwell's 1st equation: $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0 \quad\rightarrow\quad \boldsymbol{k}^{(\text{r})}\cdot\boldsymbol{E}_{\text{p}}^{(\text{r})} = 0 \quad\rightarrow\quad E_{z0}^{(\text{r})} = -(\tan\theta)E_{x0}^{(\text{r})}.$

From Maxwell's 3rd equation: $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t \quad\rightarrow\quad \boldsymbol{k}^{(\text{r})}\times\boldsymbol{E}_{\text{p}}^{(\text{r})} = \mu_0\omega\boldsymbol{H}_0^{(\text{r})}$

$$\begin{aligned}\rightarrow\quad \boldsymbol{H}_0^{(\text{r})} &= Z_0^{-1}n_0(\omega)(\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{z}})\times(E_{x0}^{(\text{r})}\hat{\boldsymbol{x}} + E_{z0}^{(\text{r})}\hat{\boldsymbol{z}})\\ &= -Z_0^{-1}n_0(\omega)[\sin\theta\,E_{z0}^{(\text{r})} - \cos\theta\,E_{x0}^{(\text{r})}]\hat{\boldsymbol{y}}\\ &= Z_0^{-1}n_0(\omega)E_{x_0}^{(\text{r})}\hat{\boldsymbol{y}}/\cos\theta\\ &= Z_0^{-1}n_0(\omega)\rho_{\text{p}}E_{x_0}^{(\text{i})}\hat{\boldsymbol{y}}/\cos\theta\\ &= Z_0^{-1}n_0(\omega)\rho_{\text{p}}E_{\text{p}}^{(\text{i})}\hat{\boldsymbol{y}}.\end{aligned}$$

Consequently, $\boldsymbol{H}^{(\text{r})}(\boldsymbol{r},t) = \boldsymbol{H}_0^{(\text{r})}\exp\left[\text{i}\left(\boldsymbol{k}^{(\text{r})}\cdot\boldsymbol{r} - \omega t\right)\right].$

b) $$\begin{aligned}\langle\boldsymbol{S}^{(\text{i})}(\boldsymbol{r},t)\rangle &= \tfrac{1}{2}\text{Re}\left\{\boldsymbol{E}_{\text{p}}^{(\text{i})}\times\boldsymbol{H}_0^{*(\text{i})}\right\} = -\tfrac{1}{2}Z_0^{-1}n_0(\omega)\text{Re}\left\{E_{\text{p}}^{(\text{i})}(\cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{z}})\times E_{\text{p}}^{*(\text{i})}\hat{\boldsymbol{y}}\right\}\\ &= \tfrac{1}{2}Z_0^{-1}n_0(\omega)\left|E_{\text{p}}^{(\text{i})}\right|^2(\sin\theta\,\hat{\boldsymbol{x}} - \cos\theta\,\hat{\boldsymbol{z}})\end{aligned}$$

$$= \tfrac{1}{2} Z_0^{-1} n_0(\omega) \left| E_{\mathrm{p}}^{(\mathrm{i})} \right|^2 \hat{\boldsymbol{k}}^{(\mathrm{i})}.$$

$$\langle \boldsymbol{S}^{(\mathrm{r})}(\boldsymbol{r},t) \rangle = \tfrac{1}{2} \mathrm{Re}\left\{ \boldsymbol{E}_{\mathrm{p}}^{(\mathrm{r})} \times \boldsymbol{H}_0^{*(\mathrm{r})} \right\} = \tfrac{1}{2} Z_0^{-1} n_0(\omega) \mathrm{Re}\left\{ \rho_{\mathrm{p}} E_{\mathrm{p}}^{(\mathrm{i})} (\cos\theta\, \hat{\boldsymbol{x}} - \sin\theta\, \hat{\boldsymbol{z}}) \times \rho_{\mathrm{p}}^* E_{\mathrm{p}}^{*(\mathrm{i})} \hat{\boldsymbol{y}} \right\}$$

$$= \tfrac{1}{2} Z_0^{-1} n_0(\omega) \left| \rho_{\mathrm{p}} E_{\mathrm{p}}^{(\mathrm{i})} \right|^2 (\sin\theta\, \hat{\boldsymbol{x}} + \cos\theta\, \hat{\boldsymbol{z}})$$

$$= \tfrac{1}{2} Z_0^{-1} n_0(\omega) |\rho_{\mathrm{p}}|^2 \left| E_{\mathrm{p}}^{(\mathrm{i})} \right|^2 \hat{\boldsymbol{k}}^{(\mathrm{r})}.$$

The time-averaged Poynting vectors of the incident and reflected beams are seen to be along the corresponding directions of propagation. The rate of flow of energy of the reflected beam is that of the incident beam multiplied by $|\rho_{\mathrm{p}}|^2$. The phase φ_{p} of the Fresnel reflection coefficient does *not* affect the reflectance of optical energy at the interface between the two media.
