Problem 7.58) a) Dispersion relation:

$$k^2 = k_x^2 + k_z^2 = (\omega/c)^2\mu_a(\omega)\varepsilon_a(\omega) \;\;\rightarrow\;\; k_z^{(\mathrm{i})} = \pm(\omega/c)\sqrt{\varepsilon_a(\omega) - (ck_x/\omega)^2}. \quad (1)$$

Since the incident wave is assumed to be evanescent, its k_z must be imaginary, and since it must decay away from the interface, only the plus sign will be acceptable. Therefore,

$$k_z^{(\mathrm{i})} = \mathrm{i}(\omega/c)\sqrt{(ck_x/\omega)^2 - \varepsilon_a(\omega)}. \quad (2)$$

Maxwell's first equation, $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$, relates E_{z0} to E_{x0}, k_x, and k_z, as follows:

$$k_x E_{x0}^{(\mathrm{i})} + k_z^{(\mathrm{i})} E_{z0}^{(\mathrm{i})} = 0 \quad\rightarrow\quad E_{z0}^{(\mathrm{i})} = -k_x E_{x0}^{(\mathrm{i})}/k_z^{(\mathrm{i})}. \quad (3)$$

Maxwell's third equation, $\boldsymbol{k}\times\boldsymbol{E}_0 = \omega\mu_0\mu(\omega)\boldsymbol{H}_0$, now yields the magnetic field, namely,

$$\boldsymbol{H}_0^{(\mathrm{i})} = \frac{\left(k_x\hat{\boldsymbol{x}} + k_z^{(\mathrm{i})}\hat{\boldsymbol{z}}\right)\times\left(E_{x0}^{(\mathrm{i})}\hat{\boldsymbol{x}} + E_{z0}^{(\mathrm{i})}\hat{\boldsymbol{z}}\right)}{\mu_0\omega} = \frac{k_z^{(\mathrm{i})}E_{x0}^{(\mathrm{i})} - k_x E_{z0}^{(\mathrm{i})}}{\mu_0\omega}\hat{\boldsymbol{y}} = \frac{k_x^2 + k_z^{(\mathrm{i})2}}{\mu_0\omega k_z^{(\mathrm{i})}}E_{x0}^{(\mathrm{i})}\hat{\boldsymbol{y}}$$

$$= \frac{(\omega/c)^2\varepsilon_a(\omega)}{\mathrm{i}\mu_0(\omega^2/c)\sqrt{(ck_x/\omega)^2 - \varepsilon_a(\omega)}}E_{x0}^{(\mathrm{i})}\hat{\boldsymbol{y}} = -\frac{\mathrm{i}\varepsilon_a(\omega)}{Z_0\sqrt{(ck_x/\omega)^2 - \varepsilon_a(\omega)}}E_{x0}^{(\mathrm{i})}\hat{\boldsymbol{y}}. \quad (4)$$

Similar calculations for the transmitted plane-wave yield

$$k_z^{(\mathrm{t})} = -\mathrm{i}(\omega/c)\sqrt{(ck_x/\omega)^2 - \varepsilon_b(\omega)}. \quad (5)$$

$$\boldsymbol{H}_0^{(\mathrm{t})} = \frac{\mathrm{i}\varepsilon_b(\omega)}{Z_0\sqrt{(ck_x/\omega)^2 - \varepsilon_b(\omega)}}E_{x0}^{(\mathrm{t})}\hat{\boldsymbol{y}}. \quad (6)$$

In the absence of a reflected wave, continuity of the tangential E- and H-fields at the boundary requires that $E_{x0}^{(\mathrm{t})} = E_{x0}^{(\mathrm{i})}$ and $H_{y0}^{(\mathrm{t})} = H_{y0}^{(\mathrm{i})}$. Therefore,

$$-\frac{\mathrm{i}\varepsilon_a(\omega)}{Z_0\sqrt{(ck_x/\omega)^2 - \varepsilon_a(\omega)}} = \frac{\mathrm{i}\varepsilon_b(\omega)}{Z_0\sqrt{(ck_x/\omega)^2 - \varepsilon_b(\omega)}} \quad\rightarrow\quad \frac{\varepsilon_a^2(\omega)}{(ck_x/\omega)^2 - \varepsilon_a(\omega)} = \frac{\varepsilon_b^2(\omega)}{(ck_x/\omega)^2 - \varepsilon_b(\omega)}$$

$$\rightarrow\;\; k_x = \left(\frac{\omega}{c}\right)\sqrt{\frac{\varepsilon_a(\omega)}{1 + [\varepsilon_a(\omega)/\varepsilon_b(\omega)]}}\,. \quad (7)$$

Note that the condition $-1 < \varepsilon_a(\omega)/\varepsilon_b(\omega) < 0$ ensures that $k_x > (\omega/c)\sqrt{\varepsilon_a(\omega)}$, which is necessary for the incident wave to be evanescent.

b) The time-averaged Poynting vector for the p-polarized plane-waves under consideration is given by

$$\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle = ½\mathrm{Re}\{\boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)] \times \boldsymbol{H}_0^*\exp[-\mathrm{i}(\boldsymbol{k}^*\cdot\boldsymbol{r} - \omega t)]\}$$

$$= ½\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\,\mathrm{Re}\left[(E_{x0}\hat{\boldsymbol{x}} + E_{z0}\hat{\boldsymbol{z}})\times H_{y0}^*\hat{\boldsymbol{y}}\right]$$

$$= ½\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\,\mathrm{Re}\left(E_{x0}H_{y0}^*\hat{\boldsymbol{z}} - E_{z0}H_{y0}^*\hat{\boldsymbol{x}}\right)$$ ← Since $E_{x0}H_{y0}^*$ is imaginary, its real part vanishes.

$$= ½\exp(-2\boldsymbol{k}''\cdot\boldsymbol{r})\,\mathrm{Re}\left[(k_x/k_z)E_{x0}H_{y0}^*\right]\hat{\boldsymbol{x}}. \quad (8)$$

For the incident wave, we have

$$\langle S_x^{(\mathrm{i})}\rangle = \frac{(ck_x/\omega)\varepsilon_a(\omega)\exp[-2(\omega/c)\sqrt{(ck_x/\omega)^2-\varepsilon_a(\omega)}\,z]}{2Z_0[(ck_x/\omega)^2-\varepsilon_a(\omega)]}|E_{x0}^{(\mathrm{i})}|^2. \tag{9}$$

Similarly, for the transmitted wave,

$$\langle S_x^{(\mathrm{t})}\rangle = \frac{(ck_x/\omega)\varepsilon_b(\omega)\exp[2(\omega/c)\sqrt{(ck_x/\omega)^2-\varepsilon_b(\omega)}\,z]}{2Z_0[(ck_x/\omega)^2-\varepsilon_b(\omega)]}|E_{x0}^{(\mathrm{t})}|^2. \tag{10}$$

Note that the energy flow direction in the dielectric is opposite to that in the metallic medium.

c) In the case of and s-polarized incident wave, we will have

$$\boldsymbol{H}_0^{(\mathrm{i})} = \frac{(k_x\hat{\boldsymbol{x}}+k_z^{(\mathrm{i})}\hat{\boldsymbol{z}})\times E_{y0}^{(\mathrm{i})}\hat{\boldsymbol{y}}}{\mu_0\omega} = (E_{y0}^{(\mathrm{i})}/Z_0)\left[(ck_x/\omega)\hat{\boldsymbol{z}} - \mathrm{i}\sqrt{(ck_x/\omega)^2-\varepsilon_a(\omega)}\,\hat{\boldsymbol{x}}\right]. \tag{11}$$

$$\boldsymbol{H}_0^{(\mathrm{t})} = \frac{(k_x\hat{\boldsymbol{x}}+k_z^{(\mathrm{t})}\hat{\boldsymbol{z}})\times E_{y0}^{(\mathrm{t})}\hat{\boldsymbol{y}}}{\mu_0\omega} = (E_{y0}^{(\mathrm{t})}/Z_0)\left[(ck_x/\omega)\hat{\boldsymbol{z}} + \mathrm{i}\sqrt{(ck_x/\omega)^2-\varepsilon_b(\omega)}\,\hat{\boldsymbol{x}}\right]. \tag{12}$$

Clearly, the tangential components of both the E-field and the H-field cannot be continuous at the interface, because, as seen in Eqs.(11) and (12), $H_{x0}^{(\mathrm{i})} \neq H_{x0}^{(\mathrm{t})}$ when $E_{y0}^{(\mathrm{i})} = E_{y0}^{(\mathrm{t})}$.