Problem 7.57)

a) Dispersion relation: $k^2 = (\omega/c)^2\mu(\omega)\varepsilon(\omega) \quad \rightarrow \quad \boldsymbol{k} = \pm(\omega/c)\sqrt{\mu(\omega)\varepsilon(\omega)}\,\hat{\boldsymbol{k}}.$ (1)

In the above expression of $\boldsymbol{k}$, both plus and minus signs for the direction of propagation are retained. Here $\hat{\boldsymbol{k}}$ is an arbitrary unit vector, and the product $\mu(\omega)\varepsilon(\omega)$ is positive.

b) Faraday's law: $\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \rightarrow \quad \mathrm{i}\boldsymbol{k} \times \boldsymbol{E}_0 = \mathrm{i}\omega\mu_0\mu(\omega)\boldsymbol{H}_0 \quad \rightarrow \quad \boldsymbol{H}_0 = \frac{\boldsymbol{k} \times \boldsymbol{E}_0}{\omega\mu_0\mu(\omega)}.$ (2)

Considering that $\mu(\omega)$ appearing in the denominator in the above expression of $\boldsymbol{H}_0$ is negative, in what follows we will write it as $-\sqrt{\mu^2(\omega)}$. We will have

$$\boldsymbol{H}_0 = \pm\frac{(\omega/c)\sqrt{\mu(\omega)\varepsilon(\omega)}}{\omega\mu_0\mu(\omega)}\,\hat{\boldsymbol{k}} \times \boldsymbol{E}_0 = \mp\frac{\sqrt{\mu(\omega)\varepsilon(\omega)}}{c\mu_0\sqrt{\mu^2(\omega)}}\,\hat{\boldsymbol{k}} \times \boldsymbol{E}_0 = \mp\frac{\hat{\boldsymbol{k}} \times \boldsymbol{E}_0}{Z_0\sqrt{\mu(\omega)/\varepsilon(\omega)}}. \quad (3)$$

c) $\langle \boldsymbol{S}(\boldsymbol{r},t)\rangle = ½\mathrm{Re}[\boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}^*(\boldsymbol{r},t)] = ½\mathrm{Re}\,\{\boldsymbol{E}_0 \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)] \times \boldsymbol{H}_0^* \exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\}$

$$= ½\mathrm{Re}(\boldsymbol{E}_0 \times \boldsymbol{H}_0^*) = \mp\frac{\mathrm{Re}[\boldsymbol{E}_0 \times (\hat{\boldsymbol{k}} \times \boldsymbol{E}_0^*)]}{2Z_0\sqrt{\mu(\omega)/\varepsilon(\omega)}} = \mp\frac{\mathrm{Re}[(\boldsymbol{E}_0\cdot\boldsymbol{E}_0^*)\hat{\boldsymbol{k}} - (\boldsymbol{E}_0\cdot\hat{\boldsymbol{k}})\boldsymbol{E}_0^*]}{2Z_0\sqrt{\mu(\omega)/\varepsilon(\omega)}}$$

$$= \mp\left[\frac{E_0'^2 + E_0''^2}{2Z_0\sqrt{\mu(\omega)/\varepsilon(\omega)}}\right]\hat{\boldsymbol{k}}. \quad (4)$$

$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$

$\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^* = (\boldsymbol{E}_0' + \mathrm{i}\boldsymbol{E}_0'')\cdot(\boldsymbol{E}_0' - \mathrm{i}\boldsymbol{E}_0'') = E_0'^2 + E_0''^2$

$\boldsymbol{E}_0 \cdot \hat{\boldsymbol{k}} = 0$ because Maxwell's first equation, $\boldsymbol{\nabla}\cdot\boldsymbol{D} = 0$, yields $\mathrm{i}\boldsymbol{k}\cdot\varepsilon_0\varepsilon(\omega)\boldsymbol{E}_0 = 0$.

Clearly, the choice of plus sign for $\boldsymbol{k}$ in Eq.(1) results in a minus sign for $\langle\boldsymbol{S}\rangle$ in Eq.(4), and vice-versa. The direction of energy flow is thus seen to be opposite that of the k-vector, the latter signifying the direction of phase propagation.

d) The Fresnel reflection coefficient from free space, where $\mu_a(\omega) = \varepsilon_a(\omega) = 1$, onto a negative-index medium having $\mu_b(\omega) = \varepsilon_b(\omega) = -1$, at normal incidence is the same for p- and s-polarized light, as follows:

$$\rho_p = \rho_s = \frac{\sqrt{\varepsilon_a/\mu_a} - \sqrt{\varepsilon_b/\mu_b}}{\sqrt{\varepsilon_a/\mu_a} + \sqrt{\varepsilon_b/\mu_b}} = \frac{1-1}{1+1} = 0. \quad (5)$$

The reflection coefficient is thus zero, because the negative-index medium is impedance matched to free space. The plane-wave transmitted into the negative-index medium must, therefore, have the same E-field and the same H-field as the incident wave, because of the required boundary conditions at the interface. From Eq.(3), we must now choose the plus sign for the H-field of the transmitted plane-wave into the negative-index medium, $\boldsymbol{H}_0 = \hat{\boldsymbol{k}} \times \boldsymbol{E}_0/Z_0$. The choice of the plus sign should also be obvious from the necessity of having the transmitted beam carry energy away from the interface, that is, $\langle\boldsymbol{S}\rangle$ and $\hat{\boldsymbol{k}}$ must be in the same direction. The choice of the plus sign for $\boldsymbol{H}_0$ then forces the k-vector in Eq.(1) to have the minus sign, that is, $\boldsymbol{k} = -(\omega/c)\hat{\boldsymbol{k}}$. The transmitted plane-wave then has the following E- and H-fields:

$$\boldsymbol{E}^{(t)}(\boldsymbol{r},t) = \boldsymbol{E}_0\exp[-\mathrm{i}(\omega/c)(\hat{\boldsymbol{k}}\cdot\boldsymbol{r} + ct)], \quad (6a)$$

$$\boldsymbol{H}^{(t)}(\boldsymbol{r},t) = (\hat{\boldsymbol{k}} \times \boldsymbol{E}_0/Z_0)\exp[-\mathrm{i}(\omega/c)(\hat{\boldsymbol{k}}\cdot\boldsymbol{r} + ct)]. \quad (6b)$$

The phase of the E- and H-fields thus travels *toward* the interface with the speed of light c.