Problem 7.56) a) Denoting the magnitude of the k-vector in free space by $k_0 = \omega/c$, we have

$$\boldsymbol{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}} = (k_0 \sin\theta\cos\phi)\hat{\boldsymbol{x}} + (k_0\sin\theta\sin\phi)\hat{\boldsymbol{y}} + (k_0\cos\theta)\hat{\boldsymbol{z}},$$

$$\boldsymbol{E}_0 = E_{x0}\hat{\boldsymbol{x}} + E_{y0}\hat{\boldsymbol{y}} + E_{z0}\hat{\boldsymbol{z}},$$

$$\boldsymbol{H}_0 = H_{x0}\hat{\boldsymbol{x}} + H_{y0}\hat{\boldsymbol{y}} + H_{z0}\hat{\boldsymbol{z}}.$$

b) $$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0 \quad\rightarrow\quad \boldsymbol{k}\cdot\boldsymbol{E}_0 = 0 \quad\rightarrow\quad k_xE_{x0} + k_yE_{y0} + k_zE_{z0} = 0$$

$$\rightarrow\quad E_{z0} = -(k_xE_{x0} + k_yE_{y0})/k_z.$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t \quad\rightarrow\quad i\boldsymbol{k}\times\boldsymbol{E}_0 = i\omega\mu_0\boldsymbol{H}_0$$

$$\rightarrow\quad \boldsymbol{H}_0 = (\mu_0\omega)^{-1}\boldsymbol{k}\times\boldsymbol{E}_0 = (\mu_0\omega)^{-1}(k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}})\times(E_{x0}\hat{\boldsymbol{x}} + E_{y0}\hat{\boldsymbol{y}} + E_{z0}\hat{\boldsymbol{z}})$$

$$\rightarrow\quad H_{x0} = (k_yE_{z0} - k_zE_{y0})/(\mu_0\omega),$$

$$H_{y0} = (k_zE_{x0} - k_xE_{z0})/(\mu_0\omega),$$

$$H_{z0} = (k_xE_{y0} - k_yE_{x0})/(\mu_0\omega).$$

The field components $E_{z0}, H_{x0}, H_{y0}, H_{z0}$ are thus determined once the components E_{x0} and E_{y0} are specified.

c) $$\begin{aligned}\boldsymbol{S}(\boldsymbol{r},t) &= \text{Re}\{\boldsymbol{E}_0\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\times\text{Re}\{\boldsymbol{H}_0\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\\
&= [\boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{E}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\times[\boldsymbol{H}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{H}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\\
&= (\boldsymbol{E}_0'\times\boldsymbol{H}_0')\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) + (\boldsymbol{E}_0''\times\boldsymbol{H}_0'')\sin^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\\
&\quad -(\boldsymbol{E}_0'\times\boldsymbol{H}_0'' + \boldsymbol{E}_0''\times\boldsymbol{H}_0')\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\\
&= ½(\boldsymbol{E}_0'\times\boldsymbol{H}_0' + \boldsymbol{E}_0''\times\boldsymbol{H}_0'') + ½(\boldsymbol{E}_0'\times\boldsymbol{H}_0' - \boldsymbol{E}_0''\times\boldsymbol{H}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\\
&\quad -½(\boldsymbol{E}_0'\times\boldsymbol{H}_0'' + \boldsymbol{E}_0''\times\boldsymbol{H}_0')\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].\end{aligned}$$

Noting that $\boldsymbol{k}$ is a real-valued vector, we will have

$$\begin{aligned}\boldsymbol{E}_0'\times\boldsymbol{H}_0' \pm \boldsymbol{E}_0''\times\boldsymbol{H}_0'' &= (\mu_0\omega)^{-1}[\boldsymbol{E}_0'\times(\boldsymbol{k}\times\boldsymbol{E}_0') \pm \boldsymbol{E}_0''\times(\boldsymbol{k}\times\boldsymbol{E}_0'')]\\
&= (\mu_0\omega)^{-1}[(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')\boldsymbol{k} - \cancelto{0}{(\boldsymbol{k}\cdot\boldsymbol{E}_0')}\boldsymbol{E}_0' \pm (\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\boldsymbol{k} \mp \cancelto{0}{(\boldsymbol{k}\cdot\boldsymbol{E}_0'')}\boldsymbol{E}_0'']\\
&= (\mu_0\omega)^{-1}(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' \pm \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\boldsymbol{k}\end{aligned}$$

$$\begin{aligned}\boldsymbol{E}_0'\times\boldsymbol{H}_0'' + \boldsymbol{E}_0''\times\boldsymbol{H}_0' &= (\mu_0\omega)^{-1}[\boldsymbol{E}_0'\times(\boldsymbol{k}\times\boldsymbol{E}_0'') + \boldsymbol{E}_0''\times(\boldsymbol{k}\times\boldsymbol{E}_0')]\\
&= 2(\mu_0\omega)^{-1}(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'')\boldsymbol{k}\end{aligned}$$

Therefore,

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{r},t) = ½(\mu_0\omega)^{-1}\{&(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'') + (\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' - \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\\
&-2(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'')\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\boldsymbol{k}.\end{aligned}$$

d) The electromagnetic momentum density in free space is given by $\boldsymbol{p}(\boldsymbol{r},t) = \boldsymbol{S}(\boldsymbol{r},t)/c^2$.

e) Assuming that $\boldsymbol{E}_0$ is real-valued, we may write the energy-density of the E-field as follows:

$$\begin{aligned}\mathcal{E}_{\mathrm{E}}(\boldsymbol{r},t) &= \tfrac{1}{2}\varepsilon_0 \mathrm{Re}\{\boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\cdot \mathrm{Re}\{\boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\} \\ &= \tfrac{1}{2}\varepsilon_0 \boldsymbol{E}_0' \cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\cdot \boldsymbol{E}_0' \cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) \\ &= \tfrac{1}{2}\varepsilon_0 \boldsymbol{E}_0'\cdot\boldsymbol{E}_0' \cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) \\ &= \tfrac{1}{4}\varepsilon_0 \boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + \tfrac{1}{4}\varepsilon_0 \boldsymbol{E}_0'\cdot\boldsymbol{E}_0' \cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].\end{aligned}$$

Similarly, the energy-density of the H-field is given by

$$\begin{aligned}\mathcal{E}_{\mathrm{H}}(\boldsymbol{r},t) &= \tfrac{1}{2}\mu_0 \mathrm{Re}\{\boldsymbol{H}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\cdot \mathrm{Re}\{\boldsymbol{H}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\} \\ &= \tfrac{1}{4}\mu_0 \boldsymbol{H}_0'\cdot\boldsymbol{H}_0' + \tfrac{1}{4}\mu_0 \boldsymbol{H}_0'\cdot\boldsymbol{H}_0' \cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].\end{aligned}$$

The above expression for the H-field energy-density may be further simplified by noting that

$$\begin{aligned}\boldsymbol{H}_0'\cdot\boldsymbol{H}_0' &= (\mu_0\omega)^{-2}(\boldsymbol{k}\times\boldsymbol{E}_0')\cdot(\boldsymbol{k}\times\boldsymbol{E}_0') = (\mu_0\omega)^{-2}[(\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0') - (\cancelto{0}{\boldsymbol{k}\cdot\boldsymbol{E}_0'})^2] \\ &= (\mu_0\omega)^{-2}(\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0') = (\varepsilon_0/\mu_0)\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'.\end{aligned}$$

It is seen that $\mathcal{E}_{\mathrm{H}}(\boldsymbol{r},t) = \mathcal{E}_{\mathrm{E}}(\boldsymbol{r},t)$. The total energy-density of the fields is, therefore, given by

$$\mathcal{E}(\boldsymbol{r},t) = \mathcal{E}_{\mathrm{E}}(\boldsymbol{r},t) + \mathcal{E}_{\mathrm{H}}(\boldsymbol{r},t) = \tfrac{1}{2}\varepsilon_0 \boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + \tfrac{1}{2}\varepsilon_0 \boldsymbol{E}_0'\cdot\boldsymbol{E}_0' \cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].$$

f) Poynting's theorem asserts that, in free space, $\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{r},t) + \partial\mathcal{E}(\boldsymbol{r},t)/\partial t = 0$. Recalling that $\boldsymbol{E}_0'' = 0$, the results obtained in parts (c) and (e) above now yield

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{r},t) &= \boldsymbol{\nabla}\cdot\{\tfrac{1}{2}(\mu_0\omega)^{-1}(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')\{1+\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\boldsymbol{k}\} \\ &= \tfrac{1}{2}(\mu_0\omega)^{-1}(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')\boldsymbol{\nabla}\cdot\left\{\{1+\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}(k_x\hat{\boldsymbol{x}}+k_y\hat{\boldsymbol{y}}+k_z\hat{\boldsymbol{z}})\right\} \\ &= -(\mu_0\omega)^{-1}(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')(k_x^2+k_y^2+k_z^2)\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)] \\ &= -\varepsilon_0\omega\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' \sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].\end{aligned}$$

$$\begin{aligned}\partial\mathcal{E}(\boldsymbol{r},t)/\partial t &= \partial\{\tfrac{1}{2}\varepsilon_0\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + \tfrac{1}{2}\varepsilon_0\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}/\partial t \\ &= \varepsilon_0\omega\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' \sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].\end{aligned}$$

The energy continuity equation (i.e., Poynting's theorem) is thus seen to be satisfied.

Digression: This problem can be solved in the general case when $\boldsymbol{E}_0'' \neq 0$, although the algebra is a bit tedious. Below we derive the energy densities of the E- and H-fields in the general case.

$$\begin{aligned}\mathcal{E}_{\mathrm{E}}(\boldsymbol{r},t) &= \tfrac{1}{2}\varepsilon_0 \mathrm{Re}\,\{\boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\cdot \mathrm{Re}\,\{\boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\} \\ &= \tfrac{1}{2}\varepsilon_0[\boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{E}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\cdot[\boldsymbol{E}_0'\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) - \boldsymbol{E}_0''\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)] \\ &= \tfrac{1}{2}\varepsilon_0[(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) + (\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\sin^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) \\ &\qquad -2(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'')\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)] \\ &= \tfrac{1}{4}\varepsilon_0\{(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' + \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'') + (\boldsymbol{E}_0'\cdot\boldsymbol{E}_0' - \boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\end{aligned}$$

$$-2\boldsymbol{E}_0'\cdot\boldsymbol{E}_0''\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}.$$

Similarly,

$$\mathcal{E}_{\text{H}}(\boldsymbol{r},t) = \tfrac{1}{2}\mu_0\text{Re}\{\boldsymbol{H}_0\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}\cdot\text{Re}\{\boldsymbol{H}_0\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}$$
$$= \tfrac{1}{4}\mu_0\{(\boldsymbol{H}_0'\cdot\boldsymbol{H}_0'+\boldsymbol{H}_0''\cdot\boldsymbol{H}_0'')+(\boldsymbol{H}_0'\cdot\boldsymbol{H}_0'-\boldsymbol{H}_0''\cdot\boldsymbol{H}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$$
$$-2\boldsymbol{H}_0'\cdot\boldsymbol{H}_0''\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}.$$

Now

$$\boldsymbol{H}_0'\cdot\boldsymbol{H}_0'\pm\boldsymbol{H}_0''\cdot\boldsymbol{H}_0'' = (\mu_0\omega)^{-2}[(\boldsymbol{k}\times\boldsymbol{E}_0')\cdot(\boldsymbol{k}\times\boldsymbol{E}_0')\pm(\boldsymbol{k}\times\boldsymbol{E}_0'')\cdot(\boldsymbol{k}\times\boldsymbol{E}_0'')]$$
$$= (\mu_0\omega)^{-2}[(\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')-\underbrace{(\boldsymbol{k}\cdot\boldsymbol{E}_0')^2}_{0}\pm(\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\mp\underbrace{(\boldsymbol{k}\cdot\boldsymbol{E}_0'')^2}_{0}]$$
$$= (\mu_0\omega)^{-2}(\boldsymbol{k}\cdot\boldsymbol{k})[(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')\pm(\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')]$$
$$= (\varepsilon_0/\mu_0)[(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0')\pm(\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')]$$

$$\boldsymbol{H}_0'\cdot\boldsymbol{H}_0'' = (\mu_0\omega)^{-2}(\boldsymbol{k}\times\boldsymbol{E}_0')\cdot(\boldsymbol{k}\times\boldsymbol{E}_0'')$$
$$= (\mu_0\omega)^{-2}[(\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'')-\underbrace{(\boldsymbol{k}\cdot\boldsymbol{E}_0')}_{0}\underbrace{(\boldsymbol{k}\cdot\boldsymbol{E}_0'')}_{0}]$$
$$= (\mu_0\omega)^{-2}(\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'')$$
$$= (\varepsilon_0/\mu_0)(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'').$$

As before, the energy densities of the *E*- and *H*-fields are seen to be equal. We will have

$$\mathcal{E}_{\text{E}}(\boldsymbol{r},t)+\mathcal{E}_{\text{H}}(\boldsymbol{r},t) = \tfrac{1}{2}\varepsilon_0\{(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'+\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')+(\boldsymbol{E}_0'\cdot\boldsymbol{E}_0'-\boldsymbol{E}_0''\cdot\boldsymbol{E}_0'')\cos[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$$
$$-2\boldsymbol{E}_0'\cdot\boldsymbol{E}_0''\sin[2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\}.$$

The Poynting vector was already derived in part (c) for the general case of $\boldsymbol{E}_0''\neq 0$. Verification of the energy continuity equation now follows the same steps as in part (f).
