Problem 7.54) a) Within the incidence medium, the x-component of the k-vector is given by $k_x=(\omega/c)\,n\sin\theta^{\text{i}}$. The Fresnel transmission coefficient τ_s thus yields the E-field amplitude transmitted into the free-space region below the prism, as follows:

$$\tau_s = E_{y\text{o}}^{\text{t}} / E_{y\text{o}}^{\text{i}} = \frac{2\mu_{\text{b}}\sqrt{\mu_{\text{a}}\varepsilon_{\text{a}}-(ck_x/\omega)^2}}{\mu_{\text{b}}\sqrt{\mu_{\text{a}}\varepsilon_{\text{a}}-(ck_x/\omega)^2}+\mu_{\text{a}}\sqrt{\mu_{\text{b}}\varepsilon_{\text{b}}-(ck_x/\omega)^2}} = \frac{2\cos\theta^{\text{i}}}{\cos\theta^{\text{i}}+\text{i}\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}}. \quad (1)$$

Note that the z-component of the evanescent field's k-vector, a purely imaginary entity, is given by

$$k_z^{\text{t}} = -\sqrt{(\omega/c)^2-k_x^2} = -(\omega/c)\sqrt{1-n^2\sin^2\theta^{\text{i}}} = -\text{i}\,(\omega/c)n\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,. \quad (2)$$

The evanescent wave's H-field may now be calculated using Maxwell's 3rd equation, namely,

$$\boldsymbol{k}\times\boldsymbol{E}_{\text{o}} = \mu_{\text{o}}\omega\boldsymbol{H}_{\text{o}} \quad\rightarrow\quad k_x E_{y\text{o}}\hat{\boldsymbol{z}} - k_z E_{y\text{o}}\hat{\boldsymbol{x}} = \mu_{\text{o}}\omega\boldsymbol{H}_{\text{o}}$$

$$\rightarrow\quad (\omega/c)n\sin\theta^{\text{i}}E_{y\text{o}}^{\text{t}}\hat{\boldsymbol{z}} + \text{i}\,(\omega/c)n\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,E_{y\text{o}}^{\text{t}}\hat{\boldsymbol{x}} = \mu_{\text{o}}\omega\boldsymbol{H}_{\text{o}}^{\text{t}}$$

$$\rightarrow\quad \boldsymbol{H}_{\text{o}}^{\text{t}} = \left[\text{i}\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,\hat{\boldsymbol{x}} + \sin\theta^{\text{i}}\hat{\boldsymbol{z}}\right] nE_{y\text{o}}^{\text{t}}/Z_{\text{o}}\,. \quad (3)$$

The complete expressions for the E- and H-fields of the evanescent wave are thus found to be

$$\boldsymbol{E}^{\text{t}}(\boldsymbol{r},t) = \text{Re}\left\{\tau_s E_{y\text{o}}^{\text{i}}\hat{\boldsymbol{y}}\exp[\text{i}(k_x x + k_z^{\text{t}} z - \omega t)]\right\}, \quad (4\text{a})$$

$$\boldsymbol{H}^{\text{t}}(\boldsymbol{r},t) = \text{Re}\left\{\left[\text{i}\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,\hat{\boldsymbol{x}} + \sin\theta^{\text{i}}\hat{\boldsymbol{z}}\right](n\tau_s E_{y\text{o}}^{\text{i}}/Z_{\text{o}})\exp[\text{i}(k_x x + k_z^{\text{t}} z - \omega t)]\right\}. \quad (4\text{b})$$

b) Noting that $\tau_s = |\tau_s|\exp(\text{i}\,\phi_{\tau_s})$, where

$$|\tau_s| = 2\cos\theta^{\text{i}}/\cos\theta_c\,, \quad (5\text{a})$$

$$\phi_{\tau_s} = -\tan^{-1}\left(\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,/\cos\theta^{\text{i}}\right), \quad (5\text{b})$$

we write the energy-density of the electromagnetic field at all points (x,y,z,t), where $z<0$, as follows:

$$\mathcal{E}(\boldsymbol{r},t) = \tfrac{1}{2}\varepsilon_{\text{o}}|\boldsymbol{E}|^2 + \tfrac{1}{2}\mu_{\text{o}}|\boldsymbol{H}|^2 = \tfrac{1}{2}|\tau_s|^2|E_{y\text{o}}^{\text{i}}|^2\exp(2\text{i}k_z^{\text{t}}z)\left\{\varepsilon_{\text{o}}\cos^2(k_x x - \omega t + \phi_{\tau_s})\right.$$
$$\left. + \mu_{\text{o}}(n^2/Z_{\text{o}}^2)[(\sin^2\theta^{\text{i}}-\sin^2\theta_c)\sin^2(k_x x - \omega t + \phi_{\tau_s}) + \sin^2\theta^{\text{i}}\cos^2(k_x x - \omega t + \phi_{\tau_s})]\right\}. \quad (6)$$

Substitution for k_z^{t} from Eq.(2) and setting $n\sin\theta_c = 1$ simplifies the above equation, yielding

$$\mathcal{E}(\boldsymbol{r},t) = \tfrac{1}{2}\varepsilon_{\text{o}}|\tau_s|^2|E_{y\text{o}}^{\text{i}}|^2\exp\left[2(\omega/c)n\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,z\right]\{\cos[2(k_x x - \omega t + \phi_{\tau_s})] + n^2\sin^2\theta^{\text{i}}\}. \quad (7)$$

Next, we calculate the Poynting vector of the evanescent field, as follows:

$$\boldsymbol{S}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)=(n/Z_{\text{o}})|\tau_s|^2|E_{y\text{o}}^{\text{i}}|^2\exp(2\mathrm{i}k_z^{\text{t}}z)\Big\{\cos(k_x x-\omega t+\phi_{\tau_s})\hat{\boldsymbol{y}}$$

$$\times\Big[-\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,\sin(k_x x-\omega t+\phi_{\tau_s})\hat{\boldsymbol{x}}+\sin\theta^{\text{i}}\cos(k_x x-\omega t+\phi_{\tau_s})\hat{\boldsymbol{z}}\Big]\Big\}. \qquad (8)$$

Substitution for k_z^{t} from Eq.(2), followed by further algebraic manipulations, simplify the above equation, yielding

$$\boldsymbol{S}(\boldsymbol{r},t)=(n/Z_{\text{o}})|\tau_s|^2|E_{y\text{o}}^{\text{i}}|^2\exp\Big[2(\omega/c)n\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,z\Big]$$

$$\times\Big\{\sin\theta^{\text{i}}\cos^2(k_x x-\omega t+\phi_{\tau_s})\hat{\boldsymbol{x}}+\tfrac{1}{2}\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,\sin[2(k_x x-\omega t+\phi_{\tau_s})]\hat{\boldsymbol{z}}\Big\}. \qquad (9)$$

To verify the energy continuity equation, we calculate its two terms separately, namely,

$$\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{r},t)=\frac{\partial S_x}{\partial x}+\frac{\partial S_z}{\partial z}=(n/Z_{\text{o}})|\tau_s|^2|E_{y\text{o}}^{\text{i}}|^2\exp\Big[2(\omega/c)n\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,z\Big]$$

$$\times\Big\{-k_x\sin\theta^{\text{i}}\sin[2(k_x x-\omega t+\phi_{\tau_s})]+(\omega/c)n(\sin^2\theta^{\text{i}}-\sin^2\theta_c)\sin[2(k_x x-\omega t+\phi_{\tau_s})]\Big\}. \qquad (10)$$

Considering that $k_x=(\omega/c)n\sin\theta^{\text{i}}$ and $n\sin\theta_c=1$, the above equation simplifies, yielding

$$\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{r},t)=-\varepsilon_{\text{o}}\omega|\tau_s|^2|E_{y\text{o}}^{\text{i}}|^2\exp\Big[2(\omega/c)n\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,z\Big]\sin[2(k_x x-\omega t+\phi_{\tau_s})]. \qquad (11)$$

Next, we calculate the time-derivative of the energy-density given by Eq.(7). We find

$$\frac{\partial\mathcal{E}(\boldsymbol{r},t)}{\partial t}=\varepsilon_{\text{o}}\omega|\tau_s|^2|E_{y\text{o}}^{\text{i}}|^2\exp\Big[2(\omega/c)n\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,z\Big]\sin[2(k_x x-\omega t+\phi_{\tau_s})]. \qquad (12)$$

It is now easy to verify that the continuity equation holds, that is, $\boldsymbol{\nabla}\cdot\boldsymbol{S}(\boldsymbol{r},t)+\partial\mathcal{E}(\boldsymbol{r},t)/\partial t=0$.

c) The time-averaged Poynting vector is readily obtained from Eq.(9), that is,

$$\langle\boldsymbol{S}(\boldsymbol{r},t)\rangle=(n/Z_{\text{o}})|\tau_s|^2|E_{y\text{o}}^{\text{i}}|^2\exp\Big[2(\omega/c)n\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,z\Big]$$

$$\times\Big\{\sin\theta^{\text{i}}\langle\cos^2(k_x x-\omega t+\phi_{\tau_s})\rangle\hat{\boldsymbol{x}}+\tfrac{1}{2}\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,\langle\sin[2(k_x x-\omega t+\phi_{\tau_s})]\rangle\hat{\boldsymbol{z}}\Big\}.$$

$$=\tfrac{1}{2}(n/Z_{\text{o}})|\tau_s|^2|E_{y\text{o}}^{\text{i}}|^2\sin\theta^{\text{i}}\exp\Big[2(\omega/c)n\sqrt{\sin^2\theta^{\text{i}}-\sin^2\theta_c}\,z\Big]\hat{\boldsymbol{x}}. \qquad (13)$$

Cleary, the time-averaged z-component of the Poynting vector is zero, whereas its x-component is a positive entity.

d) The stored areal energy-density (per unit area of the xy-plane) is obtained by integrating the time-averaged volumetric energy-density, namely, $\langle\mathcal{E}(\boldsymbol{r},t)\rangle$, along the z-axis, from $z=-\infty$ to $z=0$. We find

$$\int_{-\infty}^{0} <\mathcal{E}(x,y,z,t)> \mathrm{d}z = \tfrac{1}{2}\varepsilon_o |\tau_s|^2 |E_{yo}^{\mathrm{i}}|^2 \{<\cos[2(k_x x - \omega t + \phi_{\tau_s})]> + n^2 \sin^2\theta^{\mathrm{i}}\}$$

$$\times \int_{-\infty}^{0} \exp\left[2(\omega/c) n \sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}\, z\right] \mathrm{d}z$$

$$= \frac{\varepsilon_o n^2 \sin^2\theta^{\mathrm{i}} |\tau_s|^2 |E_{yo}^{\mathrm{i}}|^2}{4(\omega/c) n \sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}} = \frac{n (\sin 2\theta^{\mathrm{i}}/\cos\theta_c)^2 |E_{yo}^{\mathrm{i}}|^2}{4 Z_o \omega \sqrt{\sin^2\theta^{\mathrm{i}} - \sin^2\theta_c}}. \qquad (14)$$

Note that the stored energy-density increases indefinitely as θ^{i} approaches θ_c from above.