Problem 7.53) a) We use the dispersion relation to find k_z in terms of k_x, ω, and material parameters for each plane-wave. We then proceed to relate the various components of the E- and H-fields to each other and to the k-vector through the use of Maxwell's equations. The dispersion relation is

$$k^2 = k_x^2 + k_y^2 + k_z^2 = (\omega/c)^2 \mu(\omega)\varepsilon(\omega) \quad \rightarrow \quad k_z = \pm\sqrt{(\omega/c)^2\mu(\omega)\varepsilon(\omega) - k_x^2 - k_y^2}\,. \tag{1}$$

Considering that $k_y = 0$, and using the relevant parameters for each of the two media, we find

$$k_z^{\mathrm{i}} = -(\omega/c)\sqrt{\mu_{\mathrm{a}}(\omega)\,\varepsilon_{\mathrm{a}}(\omega) - (ck_x/\omega)^2}\,; \tag{2a}$$

$$k_z^{\mathrm{r}} = (\omega/c)\sqrt{\mu_{\mathrm{a}}(\omega)\varepsilon_{\mathrm{a}}(\omega) - (ck_x/\omega)^2}\,; \tag{2b}$$

$$k_z^{\mathrm{t}} = -(\omega/c)\sqrt{\mu_{\mathrm{b}}(\omega)\varepsilon_{\mathrm{b}}(\omega) - (ck_x/\omega)^2}\,; \tag{2c}$$

The choice of sign for the square root must be made such that the imaginary part of k_z is positive for upward-propagating waves, and negative for downward-propagating waves.

For p-polarized light, Maxwell's 1st equation yields

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = 0 \quad\rightarrow\quad \boldsymbol{k}\cdot\boldsymbol{E}_p = 0 \quad\rightarrow\quad k_x E_{xp} + k_z E_{zp} = 0 \quad\rightarrow\quad \begin{cases} E_{zp}^{\mathrm{i}} = -(k_x/k_z^{\mathrm{i}})E_{xp}^{\mathrm{i}} \\ E_{zp}^{\mathrm{r}} = -(k_x/k_z^{\mathrm{r}})E_{xp}^{\mathrm{r}} \\ E_{zp}^{\mathrm{t}} = -(k_x/k_z^{\mathrm{t}})E_{xp}^{\mathrm{t}} \end{cases} \tag{3}$$

As for the H-field of the various p-polarized beams, we use Maxwell's 3rd equation to write

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t \quad\rightarrow\quad \boldsymbol{k}\times\boldsymbol{E} = \mu_{\mathrm{o}}\mu(\omega)\omega\boldsymbol{H} \quad\rightarrow\quad k_z E_{xp} - k_x E_{zp} = \mu_{\mathrm{o}}\mu(\omega)\,\omega H_{yp}$$

$$\rightarrow\quad k_z E_{xp} + (k_x^2/k_z)E_{xp} = \mu_{\mathrm{o}}\mu(\omega)\omega H_{yp} \quad\rightarrow\quad (k_x^2 + k_z^2)E_{xp} = \mu_{\mathrm{o}}\mu(\omega)\,\omega k_z H_{yp}$$

$$\rightarrow\quad (\omega/c)^2\mu(\omega)\varepsilon(\omega)\,E_{xp} = \mu_{\mathrm{o}}\mu(\omega)\,\omega\, k_z H_{yp} \quad\rightarrow\quad (\omega/c)\,\varepsilon(\omega)\,E_{xp} = \mu_{\mathrm{o}} c k_z H_{yp}$$

$$\rightarrow\quad H_{yp} = \frac{(\omega/c)\varepsilon(\omega)}{Z_{\mathrm{o}}k_z}E_{xp} \quad\rightarrow\quad \begin{cases} H_{yp}^{\mathrm{i}} = \dfrac{(\omega/c)\,\varepsilon_{\mathrm{a}}(\omega)}{Z_{\mathrm{o}}k_z^{\mathrm{i}}}E_{xp}^{\mathrm{i}} \\ H_{yp}^{\mathrm{r}} = \dfrac{(\omega/c)\,\varepsilon_{\mathrm{a}}(\omega)}{Z_{\mathrm{o}}k_z^{\mathrm{r}}}E_{xp}^{\mathrm{r}} \\ H_{yp}^{\mathrm{t}} = \dfrac{(\omega/c)\,\varepsilon_{\mathrm{b}}(\omega)}{Z_{\mathrm{o}}k_z^{\mathrm{t}}}E_{xp}^{\mathrm{t}} \end{cases} \tag{4}$$

For the s-polarized light, we use Maxwell's 4th equation to relate H_z to H_x, as follows:

$$\boldsymbol{k}\cdot\boldsymbol{H} = 0 \quad\rightarrow\quad k_x H_{xs} + k_z H_{zs} = 0 \quad\rightarrow\quad \begin{cases} H_{zs}^{\mathrm{i}} = -(k_x/k_z^{\mathrm{i}})H_{xs}^{\mathrm{i}} \\ H_{zs}^{\mathrm{r}} = -(k_x/k_z^{\mathrm{r}})H_{xs}^{\mathrm{r}} \\ H_{zs}^{\mathrm{t}} = -(k_x/k_z^{\mathrm{t}})H_{xs}^{\mathrm{t}} \end{cases} \tag{5}$$

The E-field of the s-polarized beam is readily obtained from Maxwell's 2nd equation, that is,

$$\boldsymbol{\nabla}\times\boldsymbol{H}=\partial\boldsymbol{D}/\partial t \quad\rightarrow\quad \boldsymbol{k}\times\boldsymbol{H}=-\varepsilon_{\text{o}}\varepsilon(\omega)\omega\boldsymbol{E} \quad\rightarrow\quad k_z H_{xs}-k_x H_{zs}=-\varepsilon_{\text{o}}\varepsilon(\omega)\omega E_{ys}$$

$$\rightarrow\quad k_z H_{xs}+(k_x^2/k_z)H_{xs}=-\varepsilon_{\text{o}}\varepsilon(\omega)\omega E_{ys} \quad\rightarrow\quad (k_x^2+k_z^2)H_{xs}=-\varepsilon_{\text{o}}\varepsilon(\omega)\,\omega k_z E_{ys}$$

$$\rightarrow\quad (\omega/c)^2\mu(\omega)\varepsilon(\omega)H_{xs}=-\varepsilon_{\text{o}}\varepsilon(\omega)\omega k_z E_{ys} \quad\rightarrow\quad (\omega/c)\mu(\omega)H_{xs}=-\varepsilon_{\text{o}}ck_z E_{ys}$$

$$\rightarrow\quad E_{ys}=-\frac{(\omega/c)\mu(\omega)}{k_z}Z_{\text{o}}H_{xs} \quad\rightarrow\quad \begin{cases} E_{ys}^{\text{i}}=-\dfrac{(\omega/c)\,\mu_{\text{a}}(\omega)}{k_z^{\text{i}}}Z_{\text{o}}H_{xs}^{\text{i}} \\ E_{ys}^{\text{r}}=-\dfrac{(\omega/c)\,\mu_{\text{a}}(\omega)}{k_z^{\text{r}}}Z_{\text{o}}H_{xs}^{\text{r}} \\ E_{ys}^{\text{t}}=-\dfrac{(\omega/c)\,\mu_{\text{b}}(\omega)}{k_z^{\text{t}}}Z_{\text{o}}H_{xs}^{\text{t}} \end{cases} \qquad (6)$$

b) For p-polarized light, the continuity of E_x and D_z at the $z=0$ interface yields

$$\begin{cases} E_{xp}^{\text{i}}+E_{xp}^{\text{r}}=E_{xp}^{\text{t}} \\ D_{zp}^{\text{i}}+D_{zp}^{\text{r}}=D_{zp}^{\text{t}} \end{cases} \quad\rightarrow\quad \begin{cases} E_{xp}^{\text{i}}+E_{xp}^{\text{r}}=E_{xp}^{\text{t}} & (7\text{a}) \\ \varepsilon_{\text{o}}\varepsilon_{\text{a}}E_{zp}^{\text{i}}+\varepsilon_{\text{o}}\varepsilon_{\text{a}}E_{zp}^{\text{r}}=\varepsilon_{\text{o}}\varepsilon_{\text{b}}E_{zp}^{\text{t}} & (7\text{b}) \end{cases}$$

Use Eq. (3) in Eq. (7b), then substitute for E_{xp}^{t} from Eq. (7a).

$$\rightarrow\quad (\varepsilon_{\text{a}}k_x/k_z^{\text{i}})E_{xp}^{\text{i}}+(\varepsilon_{\text{a}}k_x/k_z^{\text{r}})E_{xp}^{\text{r}}=(\varepsilon_{\text{b}}k_x/k_z^{\text{t}})(E_{xp}^{\text{i}}+E_{xp}^{\text{r}})$$

$$\rightarrow\quad [(\varepsilon_{\text{a}}/k_z^{\text{r}})-(\varepsilon_{\text{b}}/k_z^{\text{t}})]E_{xp}^{\text{r}}=[(\varepsilon_{\text{b}}/k_z^{\text{t}})-(\varepsilon_{\text{a}}/k_z^{\text{i}})]E_{xp}^{\text{i}}$$

Use Eqs. (2a, 2b) to set $k_z^{\text{r}}=-k_z^{\text{i}}$.

$$\rightarrow\quad \rho_p=\frac{E_{xp}^{\text{r}}}{E_{xp}^{\text{i}}}=\frac{(\varepsilon_{\text{b}}/k_z^{\text{t}})-(\varepsilon_{\text{a}}/k_z^{\text{i}})}{(\varepsilon_{\text{a}}/k_z^{\text{r}})-(\varepsilon_{\text{b}}/k_z^{\text{t}})}=\frac{\varepsilon_{\text{a}}k_z^{\text{t}}-\varepsilon_{\text{b}}k_z^{\text{i}}}{\varepsilon_{\text{a}}k_z^{\text{t}}+\varepsilon_{\text{b}}k_z^{\text{i}}}. \qquad (8)$$

The transmission coefficient τ_p is found from Eqs. (7a) and (8), as follows:

$$\tau_p=E_{xp}^{\text{t}}/E_{xp}^{\text{i}}=1+(E_{xp}^{\text{r}}/E_{xp}^{\text{i}})=1+\rho_p=\frac{2\varepsilon_{\text{a}}k_z^{\text{t}}}{\varepsilon_{\text{a}}k_z^{\text{t}}+\varepsilon_{\text{b}}k_z^{\text{i}}}. \qquad (9)$$

c) For s-polarized light, the continuity of H_x and B_z at the $z=0$ interface yields

$$\begin{cases} H_{xs}^{\text{i}}+H_{xs}^{\text{r}}=H_{xs}^{\text{t}} \\ B_{zs}^{\text{i}}+B_{zs}^{\text{r}}=B_{zs}^{\text{t}} \end{cases} \quad\rightarrow\quad \begin{cases} H_{xs}^{\text{i}}+H_{xs}^{\text{r}}=H_{xs}^{\text{t}} & (10\text{a}) \\ \mu_{\text{o}}\mu_{\text{a}}H_{zs}^{\text{i}}+\mu_{\text{o}}\mu_{\text{a}}H_{zs}^{\text{r}}=\mu_{\text{o}}\mu_{\text{a}}H_{zs}^{\text{t}} & (10\text{b}) \end{cases}$$

Use Eq. (5) in Eq. (10b), then substitute for H_{xs}^{t} from Eq. (10a).

$$\rightarrow\quad (\mu_{\text{a}}k_x/k_z^{\text{i}})H_{xs}^{\text{i}}+(\mu_{\text{a}}k_x/k_z^{\text{r}})H_{xs}^{\text{r}}=(\mu_{\text{b}}k_x/k_z^{\text{t}})(H_{xs}^{\text{i}}+H_{xs}^{\text{r}})$$

$$\rightarrow\quad [(\mu_{\text{a}}/k_z^{\text{r}})-(\mu_{\text{b}}/k_z^{\text{t}})]H_{xs}^{\text{r}}=[(\mu_{\text{b}}/k_z^{\text{t}})-(\mu_{\text{a}}/k_z^{\text{i}})]H_{xs}^{\text{i}}$$

Use Eqs. (2a, 2b) to set $k_z^{\text{r}}=-k_z^{\text{i}}$.

$$\rightarrow\quad \frac{H_{xs}^{\text{r}}}{H_{xs}^{\text{i}}}=\frac{(\mu_{\text{b}}/k_z^{\text{t}})-(\mu_{\text{a}}/k_z^{\text{i}})}{(\mu_{\text{a}}/k_z^{\text{r}})-(\mu_{\text{b}}/k_z^{\text{t}})}=\frac{\mu_{\text{a}}k_z^{\text{t}}-\mu_{\text{b}}k_z^{\text{i}}}{\mu_{\text{a}}k_z^{\text{t}}+\mu_{\text{b}}k_z^{\text{i}}}. \qquad (11)$$

The Fresnel reflection coefficient for s-polarized light is defined as $\rho_s = E_{ys}^{\mathrm{r}} / E_{ys}^{\mathrm{i}}$. From Eq.(6), it is clear that $\rho_s = -H_{xs}^{\mathrm{r}} / H_{xs}^{\mathrm{i}}$. Therefore,

$$\rho_s = \frac{\mu_{\mathrm{b}} k_z^{\mathrm{i}} - \mu_{\mathrm{a}} k_z^{\mathrm{t}}}{\mu_{\mathrm{b}} k_z^{\mathrm{i}} + \mu_{\mathrm{a}} k_z^{\mathrm{t}}}. \tag{12}$$

The transmission coefficient for the H-field is found from Eqs.(10a) and (11), as follows:

$$H_{xs}^{\mathrm{t}} / H_{xs}^{\mathrm{i}} = 1 + (H_{xs}^{\mathrm{r}} / H_{xs}^{\mathrm{i}}) = \frac{2\mu_{\mathrm{a}} k_z^{\mathrm{t}}}{\mu_{\mathrm{a}} k_z^{\mathrm{t}} + \mu_{\mathrm{b}} k_z^{\mathrm{i}}}. \tag{13}$$

The Fresnel transmission coefficient for s-polarized light, being defined as $\tau_s = E_{ys}^{\mathrm{t}} / E_{ys}^{\mathrm{i}}$, may now be found from Eq.(6) as $\tau_s = (\mu_{\mathrm{b}} k_z^{\mathrm{i}} / \mu_{\mathrm{a}} k_z^{\mathrm{t}}) H_{xs}^{\mathrm{t}} / H_{xs}^{\mathrm{i}}$. Consequently

$$\tau_s = \frac{2\mu_{\mathrm{b}} k_z^{\mathrm{i}}}{\mu_{\mathrm{a}} k_z^{\mathrm{t}} + \mu_{\mathrm{b}} k_z^{\mathrm{i}}}. \tag{14}$$
