Problem 7.52)

a) $$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{\text{o}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)],\qquad \boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{\text{o}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].$$

Incident beam: $$k_x=(\omega/c)n\sin\theta,\qquad k_y=0,\qquad k_z^{\text{i}}=-(\omega/c)n\cos\theta.$$

$$E_{x\text{o}}^{\text{i}}=\text{arbitrary},\qquad E_{y\text{o}}^{\text{i}}=0;\qquad \boldsymbol{k}^{\text{i}}\cdot\boldsymbol{E}^{\text{i}}=0\ \rightarrow\ E_{z\text{o}}^{\text{i}}=-k_xE_{x\text{o}}^{\text{i}}/k_z^{\text{i}}.$$

$$\boldsymbol{k}^{\text{i}}\times\boldsymbol{E}^{\text{i}}=\mu_{\text{o}}\mu(\omega)\omega\boldsymbol{H}^{\text{i}}\rightarrow\ (k_x\hat{\boldsymbol{x}}+k_z^{\text{i}}\hat{\boldsymbol{z}})\times(E_{x\text{o}}^{\text{i}}\hat{\boldsymbol{x}}+E_{z\text{o}}^{\text{i}}\hat{\boldsymbol{z}})=\mu_{\text{o}}\omega H_{y\text{o}}^{\text{i}}\hat{\boldsymbol{y}}\rightarrow\ k_z^{\text{i}}E_{x\text{o}}^{\text{i}}-k_xE_{z\text{o}}^{\text{i}}=\mu_{\text{o}}\omega H_{y\text{o}}^{\text{i}}$$

$$\rightarrow H_{y\text{o}}^{\text{i}}=\frac{k_x^2+k_z^{\text{i}\,2}}{\mu_{\text{o}}\omega k_z^{\text{i}}}E_{x\text{o}}^{\text{i}}=-\frac{(\omega/c)^2n^2}{\mu_{\text{o}}\omega(\omega/c)n\cos\theta}E_{x\text{o}}^{\text{i}}=-\frac{n}{Z_{\text{o}}\cos\theta}E_{x\text{o}}^{\text{i}}.$$

Reflected beam: $$k_x=(\omega/c)n\sin\theta,\qquad k_y=0,\qquad k_z^{\text{r}}=(\omega/c)n\cos\theta.$$

$$E_{x\text{o}}^{\text{r}}=\text{unknown},\qquad E_{y\text{o}}^{\text{r}}=0;\qquad \boldsymbol{k}^{\text{r}}\cdot\boldsymbol{E}^{\text{r}}=0\ \rightarrow\ E_{z\text{o}}^{\text{r}}=-k_xE_{x\text{o}}^{\text{r}}/k_z^{\text{r}}.$$

$$\boldsymbol{k}^{\text{r}}\times\boldsymbol{E}^{\text{r}}=\mu_{\text{o}}\mu(\omega)\omega\boldsymbol{H}^{\text{r}}\rightarrow\ H_{y\text{o}}^{\text{r}}=\frac{k_x^2+k_z^{\text{r}\,2}}{\mu_{\text{o}}\omega k_z^{\text{r}}}E_{x\text{o}}^{\text{r}}=\frac{n}{Z_{\text{o}}\cos\theta}E_{x\text{o}}^{\text{r}}.$$

Evanescent beam: $$k_x=(\omega/c)n\sin\theta,\qquad k_y=0,\qquad k_z^{\text{t}}=-\mathrm{i}(\omega/c)\sqrt{n^2\sin^2\theta-1}.$$

$$E_{x\text{o}}^{\text{t}}=\text{unknown},\qquad E_{y\text{o}}^{\text{t}}=0;\qquad \boldsymbol{k}^{\text{t}}\cdot\boldsymbol{E}^{\text{t}}=0\ \rightarrow\ E_{z\text{o}}^{\text{t}}=-k_xE_{x\text{o}}^{\text{t}}/k_z^{\text{t}}.$$

$$\boldsymbol{k}^{\text{t}}\times\boldsymbol{E}^{\text{t}}=\mu_{\text{o}}\mu(\omega)\omega\boldsymbol{H}^{\text{t}}\rightarrow\ H_{y\text{o}}^{\text{t}}=\frac{k_x^2+k_z^{\text{t}\,2}}{\mu_{\text{o}}\omega k_z^{\text{t}}}E_{x\text{o}}^{\text{t}}=\frac{\mathrm{i}}{Z_{\text{o}}\sqrt{n^2\sin^2\theta-1}}E_{x\text{o}}^{\text{t}}.$$

b) Matching the boundary conditions yields the evanescent field amplitudes as follows:

$$\begin{cases}E_{x\text{o}}^{\text{i}}+E_{x\text{o}}^{\text{r}}=E_{x\text{o}}^{\text{t}}\\ H_{y\text{o}}^{\text{i}}+H_{y\text{o}}^{\text{r}}=H_{y\text{o}}^{\text{t}}\end{cases}\rightarrow\begin{cases}E_{x\text{o}}^{\text{r}}=E_{x\text{o}}^{\text{t}}-E_{x\text{o}}^{\text{i}}\\ -\dfrac{n}{Z_{\text{o}}\cos\theta}E_{x\text{o}}^{\text{i}}+\dfrac{n}{Z_{\text{o}}\cos\theta}E_{x\text{o}}^{\text{r}}=\dfrac{\mathrm{i}}{Z_{\text{o}}\sqrt{n^2\sin^2\theta-1}}E_{x\text{o}}^{\text{t}}\end{cases}$$

$$\rightarrow\ E_{x\text{o}}^{\text{t}}=\frac{2n\sqrt{n^2\sin^2\theta-1}}{n\sqrt{n^2\sin^2\theta-1}-\mathrm{i}\cos\theta}E_{x\text{o}}^{\text{i}}$$

$$E_{z\text{o}}^{\text{t}}=-k_xE_{x\text{o}}^{\text{t}}/k_z^{\text{t}}\ \rightarrow\ E_{z\text{o}}^{\text{t}}=\frac{-2\mathrm{i}n^2\sin\theta}{n\sqrt{n^2\sin^2\theta-1}-\mathrm{i}\cos\theta}E_{x\text{o}}^{\text{i}}$$

$$H_{y\text{o}}^{\text{t}}=\frac{\mathrm{i}}{Z_{\text{o}}\sqrt{n^2\sin^2\theta-1}}E_{x\text{o}}^{\text{t}}\ \rightarrow\ H_{y\text{o}}^{\text{t}}=\frac{2\mathrm{i}n\sqrt{n^2\sin^2\theta-1}}{Z_{\text{o}}\left[n\sqrt{n^2\sin^2\theta-1}-\mathrm{i}\cos\theta\right]}E_{x\text{o}}^{\text{i}}.$$

c) The time-averaged energy-density of the evanescent E-field is calculated as follows:

$$\mathcal{E}_E(\boldsymbol{r})=\tfrac{1}{2}\varepsilon_{\text{o}}<\left|\text{Re}[\boldsymbol{E}^{\text{t}}(\boldsymbol{r},t)]\right|^2>=\tfrac{1}{4}\varepsilon_{\text{o}}\text{Re}[\boldsymbol{E}^{\text{t}}(\boldsymbol{r})\cdot\boldsymbol{E}^{\text{t}*}(\boldsymbol{r})]=\tfrac{1}{4}\varepsilon_{\text{o}}\left(|E_{x\text{o}}^{\text{t}}|^2+|E_{z\text{o}}^{\text{t}}|^2\right)\exp[-2\,\text{Im}(k_z^{\text{t}})z]$$

$$= \tfrac{1}{4}\varepsilon_o \left(\left| \frac{2n\sqrt{n^2\sin^2\theta - 1}}{n\sqrt{n^2\sin^2\theta - 1} - \mathrm{i}\cos\theta} E_{xo}^{\mathrm{i}} \right|^2 + \left| \frac{-2\mathrm{i}\, n^2 \sin\theta}{n\sqrt{n^2\sin^2\theta - 1} - \mathrm{i}\cos\theta} E_{xo}^{\mathrm{i}} \right|^2 \right) \exp\!\left(2(\omega/c)\sqrt{n^2\sin^2\theta - 1}\, z\right)$$

$$\rightarrow \quad \mathcal{E}_E(\boldsymbol{r}) = \frac{n^2(2n^2\sin^2\theta - 1)}{(n^2-1)[(n^2+1)\sin^2\theta - 1]} \varepsilon_o \left| E_{xo}^{\mathrm{i}} \right|^2 \exp\!\left(2(\omega/c)\sqrt{n^2\sin^2\theta - 1}\, z\right).$$

Integration over z from $-\infty$ to 0 then yields the total E-field energy per unit area of the interface, as follows:

$$\int_{-\infty}^{0} \mathcal{E}_E(r)\,\mathrm{d}z = \frac{n^2(2n^2\sin^2\theta - 1)}{2(\omega/c)(n^2-1)[(n^2+1)\sin^2\theta - 1]\sqrt{n^2\sin^2\theta - 1}} \varepsilon_o \left| E_{xo}^{\mathrm{i}} \right|^2.$$

Similarly, the time-averaged energy-density of the evanescent H-field is calculated as follows:

$$\mathcal{E}_H(\boldsymbol{r}) = \tfrac{1}{2}\mu_o < \left|\mathrm{Re}[\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r},t)]\right|^2 > = \tfrac{1}{4}\mu_o\, \mathrm{Re}[\boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r}) \cdot \boldsymbol{H}^{\mathrm{t}*}(\boldsymbol{r})] = \tfrac{1}{4}\mu_o |H_{yo}^{\mathrm{t}}|^2 \exp[-2\,\mathrm{Im}(k_z^{\mathrm{t}}) z]$$

$$= \tfrac{1}{4}\mu_o \left| \frac{2\mathrm{i}n\sqrt{n^2\sin^2\theta - 1}}{Z_o\left[n\sqrt{n^2\sin^2\theta - 1} - \mathrm{i}\cos\theta\right]} E_{xo}^{\mathrm{i}} \right|^2 \exp\!\left(2(\omega/c)\sqrt{n^2\sin^2\theta - 1}\, z\right)$$

$$= \frac{n^2(n^2\sin^2\theta - 1)}{(n^2-1)[(n^2+1)\sin^2\theta - 1]} \varepsilon_o \left| E_{xo}^{\mathrm{i}} \right|^2 \exp\!\left(2(\omega/c)\sqrt{n^2\sin^2\theta - 1}\, z\right).$$

Integration over z from $-\infty$ to 0 then yields the total H-field energy per unit area of the interface, as follows:

$$\int_{-\infty}^{0} \mathcal{E}_H(r)\,\mathrm{d}z = \frac{n^2(n^2\sin^2\theta - 1)}{2(\omega/c)(n^2-1)[(n^2+1)\sin^2\theta - 1]\sqrt{n^2\sin^2\theta - 1}} \varepsilon_o \left| E_{xo}^{\mathrm{i}} \right|^2.$$

The total electromagnetic energy stored in the evanescent field (per unit area of the interface) may thus be obtained by adding the preceding expressions for the E- and H-field energies. Note that the energy content of the evanescent field is *not* equally split between the E- and H-fields.

If the final result is to be expressed in terms of the incident optical power P^{i} (rather than the intensity of the incident beam's E_x-component), the relation between P^{i} and $|E_{xo}^{\mathrm{i}}|^2$ is found to be

$$P^{\mathrm{i}}\hat{\boldsymbol{k}}^{\mathrm{i}} = <\boldsymbol{S}^{\mathrm{i}}(\boldsymbol{r},t)> = \tfrac{1}{2}\mathrm{Re}(\boldsymbol{E}_o^{\mathrm{i}} \times \boldsymbol{H}_o^{\mathrm{i}*}) = \tfrac{1}{2}\mathrm{Re}[(E_{xo}^{\mathrm{i}}\hat{\boldsymbol{x}} + E_{zo}^{\mathrm{i}}\hat{\boldsymbol{z}}) \times H_{yo}^{\mathrm{i}*}\hat{\boldsymbol{y}}]$$

$$= \frac{n\left|E_{xo}^{\mathrm{i}}\right|^2(\sin\theta\hat{\boldsymbol{x}} - \cos\theta\hat{\boldsymbol{z}})}{2Z_o\cos^2\theta} = \frac{n\left|E_{xo}^{\mathrm{i}}\right|^2}{2Z_o\cos^2\theta}\hat{\boldsymbol{k}}^{\mathrm{i}}.$$

Considering that the incident beam's footprint is larger than its cross-sectional area by a factor of $\cos\theta$, we may finally relate the evanescent stored energy to the incident optical power (per unit cross-sectional area of the incident beam) as follows:

$$\text{Evanescent energy} = \frac{n(3n^2\sin^2\theta - 2)\cos\theta\, P^{\mathrm{i}}}{\omega(n^2-1)[(n^2+1)\sin^2\theta - 1]\sqrt{n^2\sin^2\theta - 1}}.$$
