**Problem 7.51)**

a) $$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{\mathrm{o}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)],\qquad \boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{\mathrm{o}}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].$$

Incident beam: $$k_x=(\omega/c)\sin\theta,\qquad k_y=0,\qquad k_z^{\mathrm{i}}=-(\omega/c)\cos\theta.$$

$$E_{x\mathrm{o}}^{\mathrm{i}}=E_{z\mathrm{o}}^{\mathrm{i}}=0,\qquad E_{y\mathrm{o}}^{\mathrm{i}}=\text{arbitrary}.$$

$$\boldsymbol{k}^{\mathrm{i}}\times\boldsymbol{E}^{\mathrm{i}}=\mu_{\mathrm{o}}\mu(\omega)\omega\boldsymbol{H}^{\mathrm{i}}\rightarrow\quad(k_x\hat{\boldsymbol{x}}+k_z^{\mathrm{i}}\hat{\boldsymbol{z}})\times E_{y\mathrm{o}}^{\mathrm{i}}\hat{\boldsymbol{y}}=\mu_{\mathrm{o}}\omega(H_{x\mathrm{o}}^{\mathrm{i}}\hat{\boldsymbol{x}}+H_{z\mathrm{o}}^{\mathrm{i}}\hat{\boldsymbol{z}})$$

$$\rightarrow H_{x\mathrm{o}}^{\mathrm{i}}=\frac{\cos\theta E_{y\mathrm{o}}^{\mathrm{i}}}{Z_{\mathrm{o}}},\quad H_{z\mathrm{o}}^{\mathrm{i}}=\frac{\sin\theta E_{y\mathrm{o}}^{\mathrm{i}}}{Z_{\mathrm{o}}}.$$

Reflected beam: $$k_x=(\omega/c)\sin\theta,\qquad k_y=0,\qquad k_z^{\mathrm{r}}=(\omega/c)\cos\theta.$$

$$E_{x\mathrm{o}}^{\mathrm{r}}=E_{z\mathrm{o}}^{\mathrm{r}}=0,\qquad E_{y\mathrm{o}}^{\mathrm{r}}=\text{unknown}.$$

$$\boldsymbol{k}^{\mathrm{r}}\times\boldsymbol{E}^{\mathrm{r}}=\mu_{\mathrm{o}}\mu(\omega)\omega\boldsymbol{H}^{\mathrm{r}}\rightarrow\quad H_{x\mathrm{o}}^{\mathrm{r}}=-\frac{\cos\theta E_{y\mathrm{o}}^{\mathrm{r}}}{Z_{\mathrm{o}}},\quad H_{z\mathrm{o}}^{\mathrm{r}}=\frac{\sin\theta E_{y\mathrm{o}}^{\mathrm{r}}}{Z_{\mathrm{o}}}.$$

Transmitted beam: $$k_x=(\omega/c)\sin\theta,\qquad k_y=0,\qquad k_z^{\mathrm{t}}=-\mathrm{i}(\omega/c)\sqrt{n(\omega)^2+\sin^2\theta}\,.$$

$$E_{x\mathrm{o}}^{\mathrm{t}}=0,\qquad \boldsymbol{k}^{\mathrm{t}}\cdot\boldsymbol{E}^{\mathrm{t}}=0\quad\rightarrow\quad E_{z\mathrm{o}}^{\mathrm{t}}=-k_xE_{x\mathrm{o}}^{\mathrm{t}}/k_z^{\mathrm{t}}=0;\qquad E_{y\mathrm{o}}^{\mathrm{t}}=\text{unknown}.$$

$$\boldsymbol{k}^{\mathrm{t}}\times\boldsymbol{E}^{\mathrm{t}}=\mu_{\mathrm{o}}\mu(\omega)\omega\boldsymbol{H}^{\mathrm{t}}\rightarrow\quad H_{x\mathrm{o}}^{\mathrm{t}}=\frac{\mathrm{i}\sqrt{n(\omega)^2+\sin^2\theta}}{Z_{\mathrm{o}}}E_{y\mathrm{o}}^{\mathrm{t}},\quad H_{z\mathrm{o}}^{\mathrm{t}}=\frac{\sin\theta\,E_{y\mathrm{o}}^{\mathrm{t}}}{Z_{\mathrm{o}}}.$$

b) Matching the boundary conditions yields $E_{y\mathrm{o}}^{\mathrm{r}}$ and $E_{y\mathrm{o}}^{\mathrm{t}}$, as follows:

$$\begin{cases}E_{y\mathrm{o}}^{\mathrm{i}}+E_{y\mathrm{o}}^{\mathrm{r}}=E_{y\mathrm{o}}^{\mathrm{t}}\\ H_{x\mathrm{o}}^{\mathrm{i}}+H_{x\mathrm{o}}^{\mathrm{r}}=H_{x\mathrm{o}}^{\mathrm{t}}\end{cases}\rightarrow\quad\frac{\cos\theta E_{y\mathrm{o}}^{\mathrm{i}}}{Z_{\mathrm{o}}}-\frac{\cos\theta E_{y\mathrm{o}}^{\mathrm{r}}}{Z_{\mathrm{o}}}=\frac{\mathrm{i}\sqrt{n^2+\sin^2\theta}}{Z_{\mathrm{o}}}(E_{y\mathrm{o}}^{\mathrm{i}}+E_{y\mathrm{o}}^{\mathrm{r}})$$

$$\rightarrow\quad\frac{E_{y\mathrm{o}}^{\mathrm{r}}}{E_{y\mathrm{o}}^{\mathrm{i}}}=\frac{\cos\theta-\mathrm{i}\sqrt{n^2+\sin^2\theta}}{\cos\theta+\mathrm{i}\sqrt{n^2+\sin^2\theta}},\qquad\frac{E_{y\mathrm{o}}^{\mathrm{t}}}{E_{y\mathrm{o}}^{\mathrm{i}}}=\frac{2\cos\theta}{\cos\theta+\mathrm{i}\sqrt{n^2+\sin^2\theta}}.$$

c) The reflectance $R$ is the absolute value of the Fresnel reflection coefficient squared, that is,

$$R=\left|\frac{E_{y\mathrm{o}}^{\mathrm{r}}}{E_{y\mathrm{o}}^{\mathrm{i}}}\right|^2=\frac{\cos^2\theta+(n^2+\sin^2\theta)}{\cos^2\theta+(n^2+\sin^2\theta)}=1.$$

d) The field amplitudes decay with distance $z$ inside the plasma-like medium as $\exp[-\mathrm{Im}(k_z^{\mathrm{t}})z]$. The penetration-depth is thus a few times the inverse of $\mathrm{Im}(k_z^{\mathrm{t}})$, which is on the order of $\lambda_{\mathrm{o}}/\sqrt{n(\omega)^2+\sin^2\theta}$, where $\lambda_{\mathrm{o}}=2\pi c/\omega$ is the vacuum wavelength. Note that, if $n$ is small, the penetration-depth at normal incidence could be large, but with an increasing $\theta$, the penetration-depth shrinks rapidly. The $E$- and $H$-fields, of course, carry energy, having a total energy-density $\tfrac{1}{2}\varepsilon_o|\boldsymbol{E}^{\text{t}}|^2+\tfrac{1}{2}\mu_o|\boldsymbol{H}^{\text{t}}|^2$ inside the plasma-like medium. This energy-density must be integrated over the entire penetration-depth of the fields to yield the total energy content of the evanescent field. As for the time-averaged Poynting vector, we have

$$
\begin{aligned}
<\boldsymbol{S}^{\text{t}}(\boldsymbol{r},t)> &= \tfrac{1}{2}\operatorname{Re}[\boldsymbol{E}^{\text{t}}(\boldsymbol{r},t)\times\boldsymbol{H}^{\text{t}^*}(\boldsymbol{r},t)] \\
&= \tfrac{1}{2}\operatorname{Re}\,\{E^{\text{t}}_{y\text{o}}\hat{\boldsymbol{y}}\exp[\text{i}(\boldsymbol{k}^{\text{t}}\cdot\boldsymbol{r}-\omega t)]\times(H^{\text{t}^*}_{x\text{o}}\hat{\boldsymbol{x}}+H^{\text{t}^*}_{z\text{o}}\hat{\boldsymbol{z}})\exp[-\text{i}(\boldsymbol{k}^{\text{t}^*}\cdot\boldsymbol{r}-\omega t)]\} \\
&= \tfrac{1}{2}\exp[-2\operatorname{Im}(k^{\text{t}}_z)z]\operatorname{Re}\big(E^{\text{t}}_{y\text{o}}H^{\text{t}^*}_{z\text{o}}\hat{\boldsymbol{x}}-E^{\text{t}}_{y\text{o}}H^{\text{t}^*}_{x\text{o}}\hat{\boldsymbol{z}}\big) \\
&= \frac{\exp\Big(2(\omega/c)\sqrt{n(\omega)^2+\sin^2\theta}\;z\Big)}{2Z_\text{o}}\operatorname{Re}\Big(\sin\theta\,E^{\text{t}}_{y\text{o}}E^{\text{t}^*}_{y\text{o}}\hat{\boldsymbol{x}}+\text{i}\sqrt{n(\omega)^2+\sin^2\theta}\,E^{\text{t}}_{y\text{o}}E^{\text{t}^*}_{y\text{o}}\hat{\boldsymbol{z}}\Big) \\
&= \frac{\exp\Big(2(\omega/c)\sqrt{n(\omega)^2+\sin^2\theta}\;z\Big)}{2Z_\text{o}}\sin\theta\,\big|E^{\text{t}}_{y\text{o}}\big|^2\hat{\boldsymbol{x}}.
\end{aligned}
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Clearly, the component of the Poynting vector along the $z$-axis is zero, indicating that, in the steady-state, no electromagnetic energy is absorbed within the plasma-like medium. This is consistent with the 100% reflectivity obtained in part (c).

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