Problem 7.50) For a monochromatic plane-wave, the electromagnetic fields are written

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_{\text{o}}\exp[\text{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)], \tag{1}$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_{\text{o}}\exp[\text{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]. \tag{2}$$

In the present problem, $\boldsymbol{k}=k_x\hat{\boldsymbol{x}}+k_z\hat{\boldsymbol{z}}$, while $\boldsymbol{E}_{\text{o}}=E_{x\text{o}}\hat{\boldsymbol{x}}+E_{y\text{o}}\hat{\boldsymbol{y}}+E_{z\text{o}}\hat{\boldsymbol{z}}$ and $\boldsymbol{H}_{\text{o}}=H_{x\text{o}}\hat{\boldsymbol{x}}+H_{y\text{o}}\hat{\boldsymbol{y}}+H_{z\text{o}}\hat{\boldsymbol{z}}$. In general, the $\boldsymbol{E}$ and $\boldsymbol{H}$ components are complex-valued, k_x is real, and k_z could be real or complex. In medium 1, where μ_1 and ε_1 are real and positive and the incident wave is assumed to be homogeneous, $k_{z1}=(\omega/c)\sqrt{\mu_1\varepsilon_1-(ck_x/\omega)^2}=(\omega/c)n_1\cos\theta$ is a positive, real number. The only relevant parameter as far as medium 2 is concerned is $ck_x/\omega=n_1\sin\theta$, a real number in the range from $-n_1$ to $+n_1$.

In general, the component of the time-averaged Poynting vector along the z-axis may be written as follows:

$$<S_z(\boldsymbol{r},t)>=\tfrac{1}{2}\text{Re}(\boldsymbol{E}\times\boldsymbol{H}^*)_z=\tfrac{1}{2}\exp[-2\,\text{Im}(k_z)z]\,\text{Re}(E_{x\text{o}}H_{y\text{o}}^*-E_{y\text{o}}H_{x\text{o}}^*). \tag{3}$$

To determine $E_{x\text{o}}$ and $H_{x\text{o}}$ for substitution in the above expression, we use Maxwell's 2nd and 3rd equations, namely,

$$\boldsymbol{\nabla}\times\boldsymbol{H}=\partial\boldsymbol{D}/\partial t\;\rightarrow\;\boldsymbol{k}\times\boldsymbol{H}_{\text{o}}=-\varepsilon_{\text{o}}\varepsilon\omega\boldsymbol{E}_{\text{o}}\;\rightarrow\;(k_x\hat{\boldsymbol{x}}+k_z\hat{\boldsymbol{z}})\times(H_{x\text{o}}\hat{\boldsymbol{x}}+H_{y\text{o}}\hat{\boldsymbol{y}}+H_{z\text{o}}\hat{\boldsymbol{z}})=-\varepsilon_{\text{o}}\varepsilon\omega\boldsymbol{E}_{\text{o}}$$

$$\rightarrow k_zH_{y\text{o}}\hat{\boldsymbol{x}}+(k_xH_{z\text{o}}-k_zH_{x\text{o}})\hat{\boldsymbol{y}}-k_xH_{y\text{o}}\hat{\boldsymbol{z}}=\varepsilon_{\text{o}}\varepsilon\omega(E_{x\text{o}}\hat{\boldsymbol{x}}+E_{y\text{o}}\hat{\boldsymbol{y}}+E_{z\text{o}}\hat{\boldsymbol{z}})\rightarrow\; E_{x\text{o}}=k_zH_{y\text{o}}/(\varepsilon_{\text{o}}\varepsilon\omega). \tag{4}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t\;\rightarrow\;\boldsymbol{k}\times\boldsymbol{E}_{\text{o}}=\mu_{\text{o}}\mu\omega\boldsymbol{H}_{\text{o}}\;\rightarrow\;(k_x\hat{\boldsymbol{x}}+k_z\hat{\boldsymbol{z}})\times(E_{x\text{o}}\hat{\boldsymbol{x}}+E_{y\text{o}}\hat{\boldsymbol{y}}+E_{z\text{o}}\hat{\boldsymbol{z}})=\mu_{\text{o}}\mu\omega\boldsymbol{H}_{\text{o}}$$

$$\rightarrow -k_zE_{y\text{o}}\hat{\boldsymbol{x}}+(k_zE_{x\text{o}}-k_xE_{z\text{o}})\hat{\boldsymbol{y}}+k_xE_{y\text{o}}\hat{\boldsymbol{z}}=\mu_{\text{o}}\mu\omega(H_{x\text{o}}\hat{\boldsymbol{x}}+H_{y\text{o}}\hat{\boldsymbol{y}}+H_{z\text{o}}\hat{\boldsymbol{z}})\rightarrow\; H_{x\text{o}}=-k_zE_{y\text{o}}/(\mu_{\text{o}}\mu\omega). \tag{5}$$

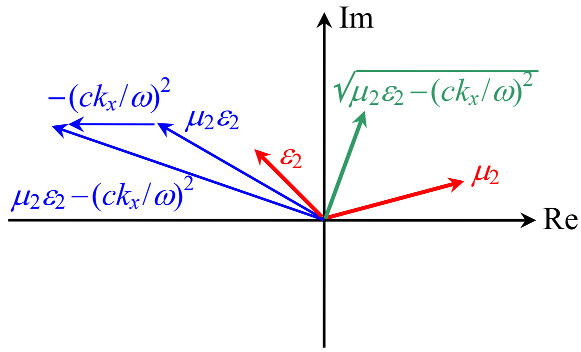
The z-component of the Poynting vector in Eq.(3) may now be written

$$<S_z(\boldsymbol{r},t)>=\tfrac{1}{2}\exp[-2\,\text{Im}(k_z)z]\,\text{Re}\left(\frac{k_zH_{y\text{o}}H_{y\text{o}}^*}{\varepsilon_{\text{o}}\varepsilon\,\omega}+\frac{k_z^*E_{y\text{o}}^*E_{y\text{o}}}{\mu_{\text{o}}\mu^*\omega}\right)$$

$$=\frac{\exp[-2\,\text{Im}(k_z)z]}{2(\omega/c)}\{Z_{\text{o}}\text{Re}(k_z/\varepsilon)|H_{y\text{o}}|^2+Z_{\text{o}}^{-1}\text{Re}(k_z/\mu)|E_{y\text{o}}|^2\}.\;\leftarrow\;\boxed{Z_{\text{o}}=\sqrt{\mu_{\text{o}}/\varepsilon_{\text{o}}}} \tag{6}$$

It is thus seen that, for p-polarized light associated with $H_{y\text{o}}$, the requirement that $<S_z>$ be positive translates into $\text{Re}(k_z/\varepsilon)\geq 0$, whereas for s-polarized light associated with $E_{y\text{o}}$, the same constraint implies that $\text{Re}(k_z/\mu)\geq 0$. In other words, for p-light, the angle between k_z and ε in the complex-plane must be less than 90º, whereas for s-light it is the angle between k_z and μ that must be below 90º.

Now, $k_{z2}=(\omega/c)\sqrt{\mu_2\varepsilon_2-(ck_x/\omega)^2}$ may be readily analyzed in the complex plane. Let us first consider the case of a lossy material, i.e., one for which both μ_2 and ε_2 are in the upper-half of the complex plane. (**Note**: μ_2 and ε_2 are slightly above the positive real axis for a transparent positive-index material, and slightly above the negative real axis for a transparent negative-index material.) With reference to the figure below, note that if k_x, which is always a real number, starts at zero and moves toward infinity, the argument of the square root starts at $\mu_2\varepsilon_2$ and moves parallel to the real axis toward $-\infty$. This means that one solution for k_{z2} starts halfway between μ_2 and ε_2 and, upon increasing k_x, moves continually toward the positive imaginary axis. It is easy to verify this result separately for cases when μ_2 and ε_2 are both in the 1st quadrant, both in the 2nd quadrant, and also when one is in the 1st and the other in the 2nd quadrant. In all cases, the solution for k_{z2} that has an acute angle with both μ_2 and ε_2 turns out to be the solution in the upper-half of the complex plane. The conclusion is that, for lossy (as well as transparent) media, the acceptable solution for k_{z2}, i.e., the solution that yields a positive $<S_z>$, is the one in the upper half-plane. The field amplitudes in medium 2 thus decay exponentially away from the interface with medium 1. Let us emphasize then, that for lossy as well as transparent media, the only acceptable solution for k_{z2} is the one that causes the electromagnetic fields to decay as they propagate along the positive z-axis. If medium 2 happens to be transparent, the exponential decay will be extremely slow below the critical angle of total internal reflection (TIR), and fast above the critical angle.



For a gain medium, both μ_2 and ε_2 will be in the lower-half of the complex plane. The same arguments as above apply, but this time the correct solution for k_{z2} is found to be in the lower half-plane. The Poynting vector component $<S_z>$ remains positive for both p- and s-polarized light, but the field amplitudes now become growing functions of z. This, of course, is understandable because the gain enables the beam amplitude to grow as it propagates away from the interface. Note that this result is valid irrespective of whether μ_2 and ε_2 are in the 3rd or 4th quadrant, and also whether the incidence angle θ is above or below the critical TIR angle.

Finally, a medium may have both loss and gain, in the sense that either ε_2 is in the upper half-plane while μ_2 is in the lower half-plane, or vice-versa. In this case, k_{z2} for p-light must be chosen to have an acute angle with ε_2, while k_{z2} for s-light must make an acute angle with μ_2. These two solutions for k_{z2} could turn out to be the same, in which case both polarization states grow or decay together as the beam propagates along z. Alternatively, it is possible for k_{z2} of p-light to be the negative of k_{z2} for s-light, in which case one polarization state grows while the other one decays as they propagate away from the interface. The exact behavior, of course, depends on the locations of μ_2 and ε_2 in the complex plane, as well as on the values of k_x and ω.