Problem 7.49) In the air-gap, where $k_{z0}=\sqrt{(\omega/c)^2-k_x^2}$, the electromagnetic (EM) fields of a TE-polarized wave are

$$\boldsymbol{E}_{\text{air}}(\boldsymbol{r},t)=E_0\hat{\boldsymbol{y}}\{\exp[\mathrm{i}(k_x x+k_{z0}z)]-\exp[\mathrm{i}(k_x x-k_{z0}z)]\}\exp(-\mathrm{i}\omega t), \tag{1a}$$

$$\boldsymbol{H}_{\text{air}}(\boldsymbol{r},t)=\frac{E_0}{\mu_0\omega}\{(k_x\hat{\boldsymbol{z}}-k_{z0}\hat{\boldsymbol{x}})\exp[\mathrm{i}(k_x x+k_{z0}z)]$$
$$-(k_x\hat{\boldsymbol{z}}+k_{z0}\hat{\boldsymbol{x}})\exp[\mathrm{i}(k_x x-k_{z0}z)]\}\exp(-\mathrm{i}\omega t). \tag{1b}$$

In the dielectric material (say, glass), where $z\ge d$ and $k_{z1}=\sqrt{\mu\varepsilon(\omega/c)^2-k_x^2}$, the TE-polarized EM fields are given by

$$\boldsymbol{E}_{\text{glass}}(\boldsymbol{r},t)=E_1\hat{\boldsymbol{y}}\exp[\mathrm{i}(k_x x+k_{z1}z)]\exp(-\mathrm{i}\omega t), \tag{2a}$$

$$\boldsymbol{H}_{\text{glass}}(\boldsymbol{r},t)=\frac{E_1}{\mu_0\mu\omega}(k_x\hat{\boldsymbol{z}}-k_{z1}\hat{\boldsymbol{x}})\exp[\mathrm{i}(k_x x+k_{z1}z)]\exp(-\mathrm{i}\omega t). \tag{2b}$$

Imposing the continuity requirement for E_y and H_x at the boundary $z=d$, we find

$$E_0[\exp(\mathrm{i}k_{z0}d)-\exp(-\mathrm{i}k_{z0}d)]=E_1\exp(\mathrm{i}k_{z1}d), \tag{3a}$$

$$\mu(\omega)E_0k_{z0}[\exp(\mathrm{i}k_{z0}d)+\exp(-\mathrm{i}k_{z0}d)]=E_1k_{z1}\exp(\mathrm{i}k_{z1}d). \tag{3b}$$

For the above equations to hold simultaneously, it is necessary that the oscillation frequency ω be a solution of the following characteristic equation:

$$\tan(k_{z0}d)=-\mathrm{i}\mu(\omega)k_{z0}/k_{z1}. \tag{4}$$

In general, the values of ω that satisfy Eq.(4) are complex, leading to leaky modes. Suppose, however, that k_x is large enough to make both k_{z0} and k_{z1} imaginary, i.e., $k_{z0}=\mathrm{i}\sqrt{k_x^2-(\omega/c)^2}$ and $k_{z1}=\mathrm{i}\sqrt{k_x^2-\mu\varepsilon(\omega/c)^2}$. Consequently, $\tan(k_{z0}d)=\mathrm{i}\tanh[\sqrt{k_x^2-(\omega/c)^2}d]$, and Eq.(4) may be rewritten as

$$\tanh\left[\sqrt{k_x^2-(\omega/c)^2}d\right]=-\frac{\mu(\omega)\sqrt{(k_x)^2-(\omega/c)^2}}{\sqrt{(k_x)^2-\mu\varepsilon(\omega/c)^2}}\,. \tag{5}$$

For a given frequency ω, when the left- and right-hand sides of Eq.(5) are plotted versus k_x [for $k_x>\sqrt{\mu\varepsilon}(\omega/c)$], the two curves will not cross at any point so long as $\mu(\omega)>0$. We conclude that the system does *not* support TE-polarized guided modes.

In the case of TM polarization, the relevant equations are similar to those already given for TE-polarized waves. We will have

$$\boldsymbol{E}_{\text{air}}(\boldsymbol{r},t)=-\frac{H_0}{\varepsilon_0\omega}\{(k_x\hat{\boldsymbol{z}}-k_{z0}\hat{\boldsymbol{x}})\exp[\mathrm{i}(k_x x+k_{z0}z)]$$
$$+(k_x\hat{\boldsymbol{z}}+k_{z0}\hat{\boldsymbol{x}})\exp[\mathrm{i}(k_x x-k_{z0}z)]\}\exp(-\mathrm{i}\omega t), \tag{6a}$$

$$\boldsymbol{H}_{\text{air}}(\boldsymbol{r},t)=H_0\hat{\boldsymbol{y}}\{\exp[\mathrm{i}(k_x x+k_{z0}z)]+\exp[\mathrm{i}(k_x x-k_{z0}z)]\}\exp(-\mathrm{i}\omega t). \tag{6b}$$

$$\boldsymbol{E}_{\text{glass}}(\boldsymbol{r},t)=-\frac{H_1}{\varepsilon_0\varepsilon\omega}(k_x\hat{\boldsymbol{z}}-k_{z1}\hat{\boldsymbol{x}})\exp[\mathrm{i}(k_x x+k_{z1}z)]\exp(-\mathrm{i}\omega t), \tag{7a}$$

$$\boldsymbol{H}_{\text{glass}}(\boldsymbol{r},t) = H_1\hat{\boldsymbol{y}}\exp[\mathrm{i}(k_x x + k_{z1}z)]\exp(-\mathrm{i}\omega t). \tag{7b}$$

Imposing the continuity requirement for E_x and H_y at the boundary $z = d$, we find

$$\varepsilon(\omega)H_0 k_{z0}[\exp(\mathrm{i}k_{z0}d) - \exp(-\mathrm{i}k_{z0}d)] = H_1 k_{z1}\exp(\mathrm{i}k_{z1}d), \tag{8a}$$

$$H_0[\exp(\mathrm{i}k_{z0}d) + \exp(-\mathrm{i}k_{z0}d)] = H_1\exp(\mathrm{i}k_{z1}d). \tag{8b}$$

For the above equations to hold simultaneously, it is necessary that the oscillation frequency ω be a solution of the following characteristic equation:

$$\varepsilon(\omega)\tan(k_{z0}d) = -\mathrm{i}k_{z1}/k_{z0}. \tag{9}$$

Once again, the general solutions for ω that satisfy Eq.(9) are complex, leading to leaky modes. Suppose, however, that k_x is large enough to make both k_{z0} and k_{z1} imaginary, in which case Eq.(9) may be rewritten as follows:

$$\tanh\left[\sqrt{k_x^2 - (\omega/c)^2}\,d\right] = -\frac{\sqrt{(k_x)^2 - \mu\varepsilon(\omega/c)^2}}{\varepsilon(\omega)\sqrt{(k_x)^2 - (\omega/c)^2}}\,. \tag{10}$$

For a given frequency ω, when the left- and right-hand sides of Eq.(10) are plotted versus k_x [for $k_x > \sqrt{\mu\varepsilon}(\omega/c)$], the two curves will not cross at any point so long as $\varepsilon(\omega) > 0$. We conclude that the system does *not* support TM-polarized guided modes either.

We mention in passing that, for a negative-index dielectric slab, in which both $\mu(\omega)$ and $\varepsilon(\omega)$ are real and negative, it might be possible to realize a guided mode in either TE or TM polarization state.
