Problem 7.43)

a) $\boldsymbol{E}(z,t)=E_{x0}\cos(k_0 z-\omega t+\varphi_0)\,\hat{\boldsymbol{x}}$ and $\boldsymbol{H}(z,t)=(E_{x0}/Z_0)\cos(k_0 z-\omega t+\varphi_0)\,\hat{\boldsymbol{y}}$. Note that $\boldsymbol{E}$ and $\boldsymbol{H}$ have the same phase φ_0.

$\boldsymbol{E}'(z,t)=E_{x0}\cos(k_0 z+\omega t+\varphi_0')\,\hat{\boldsymbol{x}}$ and $\boldsymbol{H}'(z,t)=-(E_{x0}/Z_0)\cos(k_0 z+\omega t+\varphi_0')\,\hat{\boldsymbol{y}}$. Again, $\boldsymbol{E}'$ and $\boldsymbol{H}'$ have the same phase φ_0', although it could differ from φ_0.

Total E-field: $\boldsymbol{E}(z,t)+\boldsymbol{E}'(z,t)=E_{x_0}[\cos(k_0 z-\omega t+\varphi_0)+\cos(k_0 z+\omega t+\varphi_0')]\hat{\boldsymbol{x}}$

$$=2E_{x0}\cos[k_0 z+\tfrac{1}{2}(\varphi_0'+\varphi_0)]\cos[\omega t+\tfrac{1}{2}(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{x}}.$$

Total H-field: $\boldsymbol{H}(z,t)+\boldsymbol{H}'(z,t)=(E_{x0}/Z_0)[\cos(k_0 z-\omega t+\varphi_0)-\cos(k_0 z+\omega t+\varphi_0')]\hat{\boldsymbol{y}}$

$$=2(E_{x0}/Z_0)\sin[k_0 z+\tfrac{1}{2}(\varphi_0'+\varphi_0)]\sin[\omega t+\tfrac{1}{2}(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{y}}.$$

Since the total E-field at the mirror surfaces must be zero, we will have

First mirror surface at $z=0$: $\quad 2E_{x0}\cos[\tfrac{1}{2}(\varphi_0'+\varphi_0)]\cos[\omega t+\tfrac{1}{2}(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{x}}=0.$

Second mirror surface at $z=d$: $\quad 2E_{x0}\cos[k_0 d+\tfrac{1}{2}(\varphi_0'+\varphi_0)]\cos[\omega t+\tfrac{1}{2}(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{x}}=0.$

Consequently,

$$\begin{cases}\cos[\tfrac{1}{2}(\varphi_0'+\varphi_0)]=0\\ \cos[k_0 d+\tfrac{1}{2}(\varphi_0'+\varphi_0)]=0\end{cases}\rightarrow\begin{cases}\varphi_0+\varphi_0'=(2n+1)\pi & \text{(odd multiple of }\pi);\\ k_0 d=m\pi\ \rightarrow\ d=m\lambda_0/2 & \text{(integer multiple of }\lambda_0/2).\end{cases}$$

The total fields in the cavity are thus found to be

$$\boldsymbol{E}(z,t)=2E_{x0}\sin(k_0 z)\cos[\omega t+\tfrac{1}{2}(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{x}}.$$

$$\boldsymbol{H}(z,t)=-2(E_{x0}/Z_0)\cos(k_0 z)\sin[\omega t+\tfrac{1}{2}(\varphi_0'-\varphi_0)]\,\hat{\boldsymbol{y}}.$$

b) The surface-current-density equals the total tangential H-field at each mirror surface. For the first mirror at $z=0$, $\cos(k_0 z)=1$. For the second mirror at $z=d$, $\cos(k_0 z)=\cos(m\pi)=\pm 1$. The magnitude of the surface-current-density on both mirrors is, therefore, $J_{s0}=2E_{x0}/Z_0$.

c) The trapped energy per unit cross-sectional area is given by

Trapped E-field energy:

$$\tfrac{1}{2}\varepsilon_0\int_0^d|\boldsymbol{E}(z,t)|^2\mathrm{d}z=2\varepsilon_0E_{x0}^2\cos^2[\omega t+\tfrac{1}{2}(\varphi_0'-\varphi_0)]\int_0^d\sin^2(k_0 z)\,\mathrm{d}z$$
$$=\varepsilon_0E_{x0}^2 d\cos^2[\omega t+\tfrac{1}{2}(\varphi_0'-\varphi_0)].$$

Trapped H-field energy:

$$\tfrac{1}{2}\mu_0\int_0^d|\boldsymbol{H}(z,t)|^2\mathrm{d}z=2\mu_0(E_{x0}/Z_0)^2\sin^2[\omega t+\tfrac{1}{2}(\varphi_0'-\varphi_0)]\int_0^d\cos^2(k_0 z)\,\mathrm{d}z$$
$$=\varepsilon_0E_{x0}^2 d\sin^2[\omega t+\tfrac{1}{2}(\varphi_0'-\varphi_0)].$$

The peak values of E-field and H-field energies (per unit cross-sectional area) are thus equal to $\varepsilon_0E_{x0}^2 d$. However, there exists a phase difference between these two entities: When the E-field energy is zero, the H-field energy is at a maximum, and vice-versa. At one instant, all the energy is in the E-field; a quarter of a period later, all the energy is in the H-field. The energy thus swings back and forth from one form to the other.

d) $\boldsymbol{S}(z,t) = \boldsymbol{E}(z,t) \times \boldsymbol{H}(z,t)$

$$= -(4E_{x0}^2/Z_0)\sin(k_0 z)\cos(k_0 z)\sin[\omega t + ½(\varphi_0' - \varphi_0)]\cos[\omega t + ½(\varphi_0' - \varphi_0)]\,\hat{\boldsymbol{z}}$$

$$= -(E_{x0}^2/Z_0)\sin(2k_0 z)\sin[2\omega t + (\varphi_0' - \varphi_0)]\hat{\boldsymbol{z}}.$$

At the nodes of the E-field, as well as those of the H-field, the Poynting vector is zero. No energy, therefore, crosses these nodes. In between the nodes, the energy flows to the right for one quarter of one oscillation period ($T = 2\pi/\omega$), then flows to the left during the next quarter. The process is then repeated.
