Problem 42) a) No. At $t=0$ we have $\vec{E}(\vec{r}_0,t)=\vec{E}_{0R}$, while at $t=T/4$, where T is the period of the oscillation, we'll have $\vec{E}(\vec{r}_0,t)=\vec{E}_{0I}$. If $\vec{E}_{0R}$ and $\vec{E}_{0I}$ are not aligned, the $\vec{E}$-field will have two different orientations at $t=0$ and $t=T/4$. For linear polarization, therefore, it is necessary for $\vec{E}_{0R}$ and $\vec{E}_{0I}$ to be aligned.

b) $\vec{E}(\vec{r}_0,t)=\vec{E}_{0R}\cos\omega t+\vec{E}_{0I}\sin\omega t$

When $\vec{E}_{0R}$ and $\vec{E}_{0I}$ are aligned their common direction will be the direction of $\vec{E}(\vec{r}_0,t)$. The length of $\vec{E}(\vec{r}_0,t)$ can then be written in terms of the lengths of the vectors $\vec{E}_{0R}$ and $\vec{E}_{0I}$, namely, E_{0R} and E_{0I}, as follows:

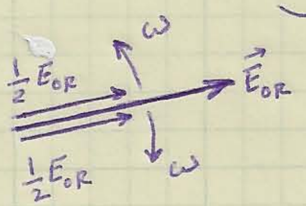
$$E(\vec{r}_0,t)=E_{0R}\cos\omega t+E_{0I}\sin\omega t=\sqrt{E_{0R}^2+E_{0I}^2}\Big(\underbrace{\frac{E_{0R}}{\sqrt{E_{0R}^2+E_{0I}^2}}}_{\cos\phi}\cos\omega t+\underbrace{\frac{E_{0I}}{\sqrt{E_{0R}^2+E_{0I}^2}}}_{\sin\phi}\sin\omega t\Big)$$

$$=\sqrt{E_{0R}^2+E_{0I}^2}\,(\cos\phi\cos\omega t+\sin\phi\sin\omega t)=\sqrt{E_{0R}^2+E_{0I}^2}\,\cos(\omega t-\phi)$$

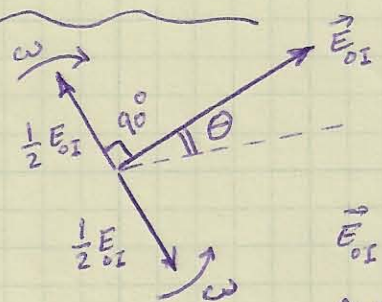
The magnitude of the $\vec{E}$-field is, therefore, $\sqrt{E_{0R}^2+E_{0I}^2}$.

c) No. At $t=0$ we have $\vec{E}(\vec{r}_0,t)=\vec{E}_{0R}$, while at $t=T/4$ we'll have $\vec{E}(\vec{r}_0,t)=\vec{E}_{0I}$. A circularly-polarized beam must have the same E-field magnitude at all times. Therefore, if $|\vec{E}_{0R}|\neq|\vec{E}_{0I}|$, the E-field magnitude at $t=0$ will differ from its magnitude at $t=T/4$, which means that the beam is not circularly polarized.

d)



$\vec{E}_{0R}\cos\omega t$ is the superposition of two circularly-polarized beams



$\vec{E}_{0I}\sin\omega t$ is also the superposition of two circularly-polarized beams.

In general, the two RCP beams combine to produce a single RCP beam. Similarly, the two LCP beams combine to produce a single LCP beam. When $\theta = \pm 90°$, however, either the two RCP beams cancel each other out, or the two LCP beams cancel each other out. The result will then be a single circularly polarized beam. When $\theta \neq 90°$, both RCP and LCP beams will have non-zero magnitudes, and the resulting beam can't be a pure circularly-polarized beam.

e) $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \Rightarrow i k_0 \vec{\sigma} \times \vec{E}_0 = -(-i\omega)\mu_0 \vec{H}_0 \Rightarrow i\dfrac{\omega}{c}\vec{\sigma} \times \vec{E}_0 = i\omega\mu_0 \vec{H}_0$

$\Rightarrow \vec{\sigma} \times \vec{E}_0 = Z_0 \vec{H}_0.$

For a homogeneous plane-wave $\vec{\sigma} = \vec{\sigma}_R$; therefore,

$$Z_0 \vec{H}_0 = \vec{\sigma}_R \times (\vec{E}_{0R} + i\vec{E}_{0I}) \Rightarrow \begin{cases} Z_0 \vec{H}_{0R} = \vec{\sigma}_R \times \vec{E}_{0R} \\ Z_0 \vec{H}_{0I} = \vec{\sigma}_R \times \vec{E}_{0I} \end{cases}$$

From Maxwell's first equation, $\vec{\nabla} \cdot \vec{E} = 0$, we know that $\vec{\sigma} \cdot \vec{E} = 0$. Therefore, for a homogeneous plane-wave, $\vec{\sigma}_R \cdot (\vec{E}_{0R} + i\vec{E}_{0I}) = 0$

$\Rightarrow \vec{\sigma}_R \cdot \vec{E}_{0R} = 0$ and $\vec{\sigma}_R \cdot \vec{E}_{0I} = 0$. In other words, the unit-vector $\vec{\sigma}_R$ is perpendicular to both $\vec{E}_{0R}$ and $\vec{E}_{0I}$. We conclude that $\vec{\sigma}_R$, $\vec{E}_{0R}$ and $\vec{H}_{0R}$ are mutually orthogonal, and also $\vec{\sigma}_R$, $\vec{E}_{0I}$ and $\vec{H}_{0I}$ are mutually orthogonal. The magnitudes of $\vec{H}_{0R}$ and $\vec{H}_{0I}$ are thus given by: $|\vec{H}_{0R}| = |\vec{E}_{0R}|/Z_0$ and $|\vec{H}_{0I}| = |\vec{E}_{0I}|/Z_0$.

f) $\vec{E}(\vec{r},t) = \mathrm{Re}\{\vec{E}_0 \exp[i(k_0\vec{\sigma}_R\cdot\vec{r} - \omega t)]\} = \vec{E}_{0R}\cos(k_0\vec{\sigma}_R\cdot\vec{r} - \omega t) - \vec{E}_{0I}\sin(k_0\vec{\sigma}_R\cdot\vec{r} - \omega t)$

$\vec{H}(\vec{r},t) = \vec{H}_{0R}\cos(k_0\vec{\sigma}_R\cdot\vec{r} - \omega t) - \vec{H}_{0I}\sin(k_0\vec{\sigma}_R\cdot\vec{r} - \omega t)$

$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t) = \vec{E}_{0R}\times\vec{H}_{0R}\cos^2(k_0\vec{\sigma}_R\cdot\vec{r} - \omega t) + \vec{E}_{0I}\times\vec{H}_{0I}\sin^2(k_0\vec{\sigma}_R\cdot\vec{r} - \omega t)$

$\qquad - (\vec{E}_{0R}\times\vec{H}_{0I} + \vec{E}_{0I}\times\vec{H}_{0R})\sin(k_0\vec{\sigma}_R\cdot\vec{r} - \omega t)\cos(k_0\vec{\sigma}_R\cdot\vec{r} - \omega t)$

Using the vector identity $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$, we write:

$$\vec{E}_{0R}\times\vec{H}_{0R} = \frac{1}{Z_0}\vec{E}_{0R}\times(\vec{\sigma}_R\times\vec{E}_{0R}) = \frac{1}{Z_0}(\vec{E}_{0R}\cdot\vec{E}_{0R})\vec{\sigma}_R - \frac{1}{Z_0}\cancelto{0}{(\vec{E}_{0R}\cdot\vec{\sigma}_R)}\vec{E}_{0R}$$

$$\vec{E}_{0I}\times\vec{H}_{0I} = \frac{1}{Z_0}\vec{E}_{0I}\times(\vec{\sigma}_R\times\vec{E}_{0I}) = \frac{1}{Z_0}(\vec{E}_{0I}\cdot\vec{E}_{0I})\vec{\sigma}_R - \frac{1}{Z_0}\cancelto{0}{(\vec{E}_{0I}\cdot\vec{\sigma}_R)}\vec{E}_{0I}$$

$$\vec{E}_{0R}\times\vec{H}_{0I} = \frac{1}{Z_0}\vec{E}_{0R}\times(\vec{\sigma}_R\times\vec{E}_{0I}) = \frac{1}{Z_0}(\vec{E}_{0R}\cdot\vec{E}_{0I})\vec{\sigma}_R - \frac{1}{Z_0}\cancelto{0}{(\vec{E}_{0R}\cdot\vec{\sigma}_R)}\vec{E}_{0I}$$

$$\vec{E}_{0I}\times\vec{H}_{0R} = \frac{1}{Z_0}\vec{E}_{0I}\times(\vec{\sigma}_R\times\vec{E}_{0R}) = \frac{1}{Z_0}(\vec{E}_{0I}\cdot\vec{E}_{0R})\vec{\sigma}_R - \frac{1}{Z_0}\cancelto{0}{(\vec{E}_{0I}\cdot\vec{\sigma}_R)}\vec{E}_{0R}$$

Therefore,

$$\vec{S}(\vec{r},t) = \frac{\vec{\sigma}_R}{Z_0}\Big\{\vec{E}_{0R}\cdot\vec{E}_{0R}\cos^2(k_0\vec{\sigma}_R\cdot\vec{r}-\omega t) + \vec{E}_{0I}\cdot\vec{E}_{0I}\sin^2(k_0\vec{\sigma}_R\cdot\vec{r}-\omega t) - \vec{E}_{0R}\cdot\vec{E}_{0I}\sin[2(k_0\vec{\sigma}_R\cdot\vec{r}-\omega t)]\Big\} \Rightarrow$$

$$\vec{S}(\vec{r},t) = \frac{\vec{\sigma}_R}{2Z_0}\Big\{(|\vec{E}_{0R}|^2+|\vec{E}_{0I}|^2) + (|\vec{E}_{0R}|^2-|\vec{E}_{0I}|^2)\cos(2k_0\vec{\sigma}_R\cdot\vec{r}-2\omega t) - 2\vec{E}_{0R}\cdot\vec{E}_{0I}\sin(2k_0\vec{\sigma}_R\cdot\vec{r}-2\omega t)\Big\}$$

Note that, for circularly polarized beams, $|\vec{E}_{0R}| = |\vec{E}_{0I}|$ and $\vec{E}_{0R}\cdot\vec{E}_{0I} = 0$; therefore, $\vec{S}(\vec{r},t) = \langle\vec{S}(\vec{r},t)\rangle$, that is, the rate of flow of energy is independent of $\vec{r}$ and t.