Problem 41)

a) $\vec{E}_1(z,t) = E_{0x}\cos(k_1 z - \omega_1 t + \phi_1)\,\hat{x}$

$$\vec{H}_1(z,t) = \frac{E_{0x}}{Z_0}\cos(k_1 z - \omega_1 t + \phi_1)\,\hat{y}$$

$$\vec{E}_2(z,t) = E_{0x}\cos(k_2 z + \omega_2 t - \phi_2)\,\hat{x}$$

$$\vec{H}_2(z,t) = \frac{-E_{0x}}{Z_0}\cos(k_2 z + \omega_2 t - \phi_2)\,\hat{y}$$

b) $\vec{E}(z,t) = \vec{E}_1 + \vec{E}_2 = E_{0x}\,\hat{x}\left\{\cos(k_1 z - \omega_1 t + \phi_1) + \cos(k_2 z + \omega_2 t - \phi_2)\right\}$

$$= 2E_{0x}\,\hat{x}\cos\left[\tfrac{1}{2}(k_1+k_2)z + \tfrac{1}{2}(\omega_2-\omega_1)t + \tfrac{1}{2}(\phi_1-\phi_2)\right]\cos\left[\tfrac{1}{2}(k_1-k_2)z - \tfrac{1}{2}(\omega_1+\omega_2)t + \tfrac{1}{2}(\phi_1+\phi_2)\right]$$

$$= 2E_{0x}\,\hat{x}\cos\left[\frac{\omega_0 z}{c} + \frac{1}{2}\Delta\omega t + \frac{\phi_1-\phi_2}{2}\right]\cos\left[\frac{\Delta\omega z}{2c} + \omega_0 t - \frac{\phi_1+\phi_2}{2}\right]$$

$$\vec{H}(z,t) = \vec{H}_1 + \vec{H}_2 = \frac{E_{0x}}{Z_0}\,\hat{y}\left\{\cos(k_1 z - \omega_1 t + \phi_1) - \cos(k_2 z + \omega_2 t - \phi_2)\right\}$$

$$= +2\frac{E_{0x}}{Z_0}\,\hat{y}\sin\left[\frac{\omega_0 z}{c} + \frac{1}{2}\Delta\omega t + \frac{\phi_1-\phi_2}{2}\right]\sin\left[\frac{\Delta\omega z}{2c} + \omega_0 t - \frac{\phi_1+\phi_2}{2}\right]$$

c) E-field energy density $= \frac{1}{2}\epsilon_0 |E|^2 =$

$$2\epsilon_0 E_{0x}^2 \cos^2\left(\frac{\omega_0 z}{c} + \frac{1}{2}\Delta\omega t + \frac{\varphi_1 - \varphi_2}{2}\right)\cos^2\left(\frac{\Delta\omega z}{2c} + \omega_0 t - \frac{\varphi_1 + \varphi_2}{2}\right)$$

The time-averaged energy density over one period of rapid oscillation (i.e., $T = 2\pi/\omega_0$) may be obtained by ignoring the envelope fluctuations, as the envelope oscillates with the much lower frequency of $\Delta\omega$. Therefore,

$$\langle \text{E-field energy density} \rangle \simeq \epsilon_0 E_{0x}^2 \cos^2\left[\frac{\omega_0}{c}\left(z + \frac{c\Delta\omega}{2\omega_0}t\right) + \frac{\varphi_1 - \varphi_2}{2}\right]$$

Similarly,

$$\langle \text{H-field energy density} \rangle = \frac{1}{2}\mu_0 \langle H^2 \rangle \simeq \epsilon_0 E_{0x}^2 \sin^2\left[\frac{\omega_0}{c}\left(z + \frac{c\Delta\omega}{2\omega_0}t\right) + \frac{\varphi_1 - \varphi_2}{2}\right]$$

The energy densities have a period of $\lambda_0/2$ along the z-axis, are shifted (relative to each other) by $\lambda_0/4$, and travel in the same direction with a speed $v = \frac{c\Delta\omega}{2\omega_0}$. The direction of travel depends on the sign of $\Delta\omega = \omega_2 - \omega_1$; in general, the fringes travel in the direction of the beam that has the higher frequency. The <u>sum</u> of the two (time-averaged) energy densities, however, is constant and stationary.

d) $\vec{S} = \vec{E} \times \vec{H} = \frac{E_{0x}^2}{Z_0}\hat{z}\sin\left[\frac{2\omega_0}{c}\left(z + \frac{c\Delta\omega}{2\omega_0}t\right) + \varphi_1 - \varphi_2\right]\sin\left[\frac{\Delta\omega z}{c} + 2\omega_0 t - (\varphi_1 + \varphi_2)\right]$

The rapid oscillations of the second sinusoid (frequency $= 2\omega_0$) yield a time-averaged value of zero for $\vec{S}$. Therefore, $\langle S_z \rangle = 0$; the energy does <u>not</u> flow in either direction.

A particularly interesting case occurs when $\Delta\omega = 0$. Here the fringes are stationary, and the Poynting vector $\vec{S}(r,t)$ shows how the energy is exchanged between the $\vec{E}$- and $\vec{H}$-fields.