Problem 40) a, b) Incident beam: $\sigma^{(i)} = (n_0 \sin\theta,\ 0,\ -n_0 \cos\theta)$

$$Z_0 H_0^{(i)} = \sigma^{(i)} \times E_0^{(i)} = n_0(\sin\theta\,\hat{x} - \cos\theta\,\hat{z}) \times E_{oy}^{(i)}\hat{y} \Rightarrow H_0^{(i)} = \frac{n_0}{Z_0}(\cos\theta\,\hat{x} + \sin\theta\,\hat{z})E_{oy}^{(i)}$$

Reflected beam: $\sigma^{(r)} = (n_0 \sin\theta,\ 0,\ n_0 \cos\theta)$

$$Z_0 H_0^{(r)} = \sigma^{(r)} \times E_0^{(r)} = n_0(\sin\theta\,\hat{x} + \cos\theta\,\hat{z}) \times E_{oy}^{(r)}\hat{y} \Rightarrow H_0^{(r)} = \frac{n_0}{Z_0}(-\cos\theta\,\hat{x} + \sin\theta\,\hat{z})\, r E_{oy}^{(i)}$$

Beam $\underline{a}$ in the gap: $\sigma^{(a)} = (n_0 \sin\theta,\ 0,\ -i\sqrt{n_0^2 \sin^2\theta - 1}\,) \leftarrow$ Note: $\theta > \theta_c$ means that $n_0 \sin\theta > 1$

$$Z_0 H_0^{(a)} = \sigma^{(a)} \times E_0^{(a)} = (n_0 \sin\theta\,\hat{x} - i\sqrt{n_0^2 \sin^2\theta - 1}\,\hat{z}) \times E_{oy}^{(a)}\hat{y} \Rightarrow$$

$$H_0^{(a)} = \frac{1}{Z_0}\left(i\sqrt{n_0^2 \sin^2\theta - 1}\,\hat{x} + n_0 \sin\theta\,\hat{z}\right) a E_{oy}^{(i)}$$

$-i$ in front of $\sigma_z^{(a)}$ means the beam decays along the negative z-axis.

Beam $\underline{b}$ in the gap: $\sigma^{(b)} = (n_0 \sin\theta,\ 0,\ +i\sqrt{n_0^2 \sin^2\theta - 1}\,)$

$$Z_0 H_0^{(b)} = \sigma^{(b)} \times E_0^{(b)} = (n_0 \sin\theta\,\hat{x} + i\sqrt{n_0^2 \sin^2\theta - 1}\,\hat{z}) \times E_{oy}^{(b)}\hat{y} \Rightarrow$$

$$H_0^{(b)} = \frac{1}{Z_0}\left(-i\sqrt{n_0^2 \sin^2\theta - 1}\,\hat{x} + n_0 \sin\theta\,\hat{z}\right) b E_{oy}^{(i)}$$

Transmitted beam: $\sigma^{(t)} = (n_0 \sin\theta,\ 0,\ -n_0 \cos\theta)$

$$Z_0 H_0^{(t)} = \sigma^{(t)} \times E_0^{(t)} = \underbrace{(n_0 \sin\theta\,\hat{x} - n_0 \cos\theta\,\hat{z})}\times E_{oy}^{(t)}\hat{y} \Rightarrow H_0^{(t)} = \frac{n_0}{Z_0}(\cos\theta\,\hat{x} + \sin\theta\,\hat{z})\, \tau E_{oy}^{(i)}$$

c) Continuity equations at $z = 0$:

E_y-field: $E_{oy}^{(i)} + E_{oy}^{(r)} = E_{oy}^{(a)} + E_{oy}^{(b)} \Rightarrow 1 + r = a + b$

H_x-field: $\frac{n_0}{Z_0}\cos\theta\, E_{oy}^{(i)} - \frac{n_0}{Z_0}\cos\theta\, E_{oy}^{(r)} = \frac{i\sqrt{n_0^2 \sin^2\theta - 1}}{Z_0} E_{oy}^{(a)} - \frac{i\sqrt{n_0^2 \sin^2\theta - 1}}{Z_0} E_{oy}^{(b)}$

$$\Rightarrow 1 - r = \frac{i\sqrt{n_0^2 \sin^2\theta - 1}}{n_0 \cos\theta}(a - b)$$

Continuity equations at $z=-d$:

E_y-field: $E_{oy}^{(a)} e^{-ik_o\sigma_z^{(a)}d} + E_{oy}^{(b)} e^{-ik_o\sigma_z^{(b)}d} = E_{oy}^{(t)} e^{-ik_o\sigma_z^{(t)}d} \Rightarrow$

$$a e^{-k_o d\sqrt{n_o^2\sin^2\theta-1}} + b e^{+k_o d\sqrt{n_o^2\sin^2\theta-1}} = \tau e^{ik_o n_o d\cos\theta}$$

H_x-field: $\frac{i\sqrt{n_o^2\sin^2\theta-1}}{Z_o} E_{oy}^{(a)} e^{-k_o d\sqrt{n_o^2\sin^2\theta-1}} - \frac{i\sqrt{n_o^2\sin^2\theta-1}}{Z_o} E_{oy}^{(b)} e^{+k_o d\sqrt{n_o^2\sin^2\theta-1}}$

$$= \frac{n_o\cos\theta}{Z_o} E_{oy}^{(t)} e^{ik_o n_o d\cos\theta}$$

$$\Rightarrow \frac{i\sqrt{n_o^2\sin^2\theta-1}}{n_o\cos\theta}\left(a e^{-k_o d\sqrt{n_o^2\sin^2\theta-1}} - b e^{+k_o d\sqrt{n_o^2\sin^2\theta-1}}\right) = \tau e^{ik_o n_o d\cos\theta}$$

d) Let $c = \frac{i\sqrt{n_o^2\sin^2\theta-1}}{n_o\cos\theta}$. Then the first two equations can be solved for $\underline{a}$ and $\underline{b}$ in terms of $\underline{r}$, as follows:

$$\begin{cases} a+b = 1+r \\ a-b = \frac{1}{c}(1-r) \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2}(1+\frac{1}{c}) + \frac{r}{2}(1-\frac{1}{c}) \\ b = \frac{1}{2}(1-\frac{1}{c}) + \frac{r}{2}(1+\frac{1}{c}) \end{cases}$$

The 3rd and 4th equations yield a relation between $\underline{a}$ and $\underline{b}$ when their left-hand-sides are set equal to each other:

$$a e^{-k_o d\sqrt{\cdots}} + b e^{+k_o d\sqrt{\cdots}} = c\left(a e^{-k_o d\sqrt{\cdots}} - b e^{+k_o d\sqrt{\cdots}}\right) \Rightarrow$$

$$a e^{-k_o d\sqrt{\cdots}}(c-1) = b e^{+k_o d\sqrt{\cdots}}(c+1) \Rightarrow$$

$$\left[\frac{c+1}{\cancel{2c}} + \frac{r(c-1)}{\cancel{2c}}\right] e^{-2k_o d\sqrt{n_o^2\sin^2\theta-1}}(c-1) = \left[\frac{c-1}{\cancel{2c}} + \frac{r(c+1)}{\cancel{2c}}\right](c+1) \Rightarrow$$

$$\left[(c^2-1) + r(c-1)^2\right] e^{-2k_o d\sqrt{n_o^2\sin^2\theta-1}} = (c^2-1) + r(c+1)^2 \Rightarrow$$

$$(c^2-1)(1-e^{-2k_o d\sqrt{\cdots}}) = r\left[(c-1)^2 e^{-2k_o d\sqrt{\cdots}} - (c+1)^2\right] \Rightarrow$$

$$r = \frac{1 - \exp\left(-2k_0 d\sqrt{n_0^2 \sin^2\theta - 1}\right)}{\left(\frac{c-1}{c+1}\right)\exp\left(-2k_0 d\sqrt{n_0^2\sin^2\theta - 1}\right) - \left(\frac{c+1}{c-1}\right)}$$

$\leftarrow k_0 = \frac{2\pi}{\lambda_0}$

In the above equation $\frac{c-1}{c+1} = \frac{i\sqrt{n_0^2\sin^2\theta - 1} - n_0\cos\theta}{i\sqrt{n_0^2\sin^2\theta - 1} + n_0\cos\theta}$ is the Fresnel reflection coefficient for s-polarized light in the limit when $d \to \infty$. Once r is determined, one can substitute it in the preceding equations for $\underline{a}$ and $\underline{b}$ to determine these coefficients. Finally, $\underline{a}$ and $\underline{b}$ can be put into either the 3rd or the 4th continuity equation to determine τ.